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Plana. III. 87

Molto Revd.° Pre Sig. Sig. Prone Colmo

Il Sig. Waring nel suo libro _Miscellanea_ _Analytica_ a pag. 36, dove applica i suoi principii alla risoluzione delle equazioni del 5.° grado, fa al fondo della pagina di proprio carattere questa breve nota = Hæc regula fallit, si summa duarum vel q: c: et summa trium radicum nihilo sit æqualis. Io non intendo quell'abbreviatura q: c: Ho procurato di trascriverla tal quale si trova: ma non giurerei, che la seconda lettera fosse veramente c. Desidero, che questa postilla così com'è possa servirle. Non ho ancor veduta la sua dissertazione idrodinamica, che ha riportato il premio di Mantova, e dall'altra parte sono voglioso assai di vederla, massimamente essendo cosa sua. Accetto dunque ben volentieri la gentile esibizione, che mi fa, di favorirmela per comoda occasione; e fin da quest'ora la ringrazio con tutto l'animo. La stampa del sesto tomo degli atti dell'Accad.a si trova finalmente a buon termine. Gradirò sommamente per il settimo tomo, che assolutamente voglio, che tardi poco ad uscire dopo il sesto, le memorie, che Ella mi accenna. Saranno uno dei principali ornamenti del tomo. Si assicuri della perfetta mia stima, e devota servitù; e vivamente raccomandandomi alla sua buona grazia ed amicizia rispettosamente mi protesto

di V. R.

Bologna li 22 Ott.e 1776

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R. DELLE SCIENZE
DI TORINO

Devotmo Servo ed Obbligmo amico
Sebastiano Canterzani

Al Molto Revd.o Pre Sig.r Sig.r Prone Col.mo
Il Pre D. Gregorio Fontana delle Scuole Pie
pub.co Profess.e di Matem.ca nella Regia Univer.tà di

Pavia

MISCELLANEA ANALYTICA,

DE

ÆQUATIONIBUS ALGEBRAICIS,

ET

CURVARUM PROPRIETATIBUS.

AB

EDVARDO WARING, A.M.

MAGD. COLL. CANTAB. SOCIO,

ET

MATHESEOS PROFESSORE LUCASIANO.

CANTABRIGIÆ,

TYPIS ACADEMICIS EXCUDEBAT J. BENTHAM.

Veneunt apud GUL. THURLBOURN & J. WOODYER, et T. & J. MERRILL, *Cantabrigiæ*; R. & J. DODSLEY, J. BEECROFT, J. WHISTON & B. WHITE, et B. DOD *Londini*; J. FLETCHER *Oxoniæ*; J. COTTON *Salopiæ*; HAMILTON & BALFOUR *Edinburgiæ*; FOULIS *Glaſguiæ*; et EWING *Dublinii.*

M.DCC.LXII.

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&c. &c. &c.

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HAS SUAS

IN

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OMNI OBSERVANTIA

D.D.D.

EDVARDUS WARING.

PRÆFATIO.

ARS *Analytica in Propoſitionibus demonſtrandis non ſolùm verſatur, ſed etiam methodum docet ipſas inveniendi; huic debetur illa propoſitionum, præcipuè univerſalium, quas invenêre Recentiores, pæne infinita multitudo. De hujùſce Scientiæ Progreſſu, et quæ diverſis temporibus acceperit, incrementis, abs re haud alienum erit pauca præfari.*

Cubicarum Æquationum reſolutionem docuit SCIPIO FERREUS; *Biquadraticas reſolvendi methodos diverſas invenêre* VIETA *et* CARTESIUS. *Coefficientem ſecundi Æquationis termini ſummam eſſe radicum, et ſimiliter omnium coefficientium conſtitutionem docuit, etiamque Radices datæ Æquationis per datas Quantitates auxit et diminuit, multiplicavit et diviſit ille hodiernæ Algebræ Reſtaurator* VIETA.

Originem Æquationum ſuperiorum ex compoſitione lateralium, et exinde Coefficientium Conſtitutionem oſtendit ille magnus Algebraiſta HARRIOTTUS.

Tot Radices affirmativas eſſe, quot ſunt mutationes Signorum a + *in* —; *cæteras vero negativas eſſe, dixit* CARTESIUS.

Methodum extrahendi Radicem quadraticam ex qüibuſcunque binomiis ſurdis dedit OUGHTREDUS; *generalem methodum extrahendi quamlibet Radicem tradidit* SCHOOTENIUS; *datæ Æquationis ſimplices Diviſores invenire docuit* JACOBUS à WASSENAER; *Quadraticos, Cubicos, &c. inveniendi methodum tradidit* HUDDENIUS: *invenit etiam methodum detegendi, utrum duæ vel plures datæ Æquationis Radices ſunt inter ſe æquales. Duas vel plures Æquationes in unam, ita ut incognitæ Quantitates exterminentur, diverſis modis transformavêre* ERASMUS BARTHOLINUS *et* CRAMER; *etiamque oſtendit* CRAMER *quomodo Quantitates utcunque ſurdæ poſſunt ex Æquationibus tolli.*

Ille

Ille Philoſophorum Princeps, et arte Analyticâ maximè inſignis NEWTONUS, *regulam dedit, ex quâ inveniri poteſt Summa datæ Æquationis Radicum, Summa Quadratorum ex ſingulis Radicibus, Cuborum, &c. invenit etiam Regulam, ex quâ ſæpe cognoſci poteſt, quot datæ Æquationis Radices ſunt poſſibiles, et quot impoſſibiles; et multas Æquationes* (2n) *Dimenſionum in Æquationes* (n) *Dimenſionum reduxit.*

Hìc non loquor de ejus eximio binomiali Theoremate, vel infinitis Seriebus: in Fluxionalem Partem hæc transfero.

WALLISIUS *deduxit Methodum, quâ integri Valores datæ Æquationis Radicum, duas incognitas Quantitates habentis, inveniri poſſunt.*

De aliorum Inventionibus ſatis dictum eſt; jam quæ in hoc Tractatu contineantur, breviter ſunt recenſenda.

In primo Capite docetur Methodus inveniendi Æquationem, cujus Radices quamcunque datam Relationem habent ad Radices datæ Æquationis vel datarum Æquationum.

In Secundo agitur de impoſſibilibus Radicibus.

In Tertio dantur Methodi detegendi datæ Æquationis Radicum rationales Valores, etiamque Radices datam inter ſe habentes Relationem.

In Quarto tractatur de Reductione & Reſolutione Æquationum.

In Quinto de duabus vel pluribus Æquationibus duas vel plures incognitas Quantitates habentibus.

Hactenus de primæ hujuſce Operis Partis Subjecto, nunc de Curvarum Proprietatibus.

Veteres, inter quos longè eminet APOLLONIUS, *Conicas Sectiones præcipuè tractârunt; Recentiores verò novos aperuerunt Curvarum Campos in infinitum diffuſos;* D' NICHOLE *Curvas ſuper Baſes ſibi æquales rotavit;* CARTESIUS *diſtinxit Algebraicas Curvas in ordines ſecundum Dimenſiones Æquationum Relationem inter abſciſſas & earum correſpondentes ordinatas exprimentium.*

Hanc Diſciplinam multum auxit Doctiſſimus BARROW; *docuit Tangentes, Aſymptotos, &c. inveniendi Methodos.*

Ille

Ille præclarus Geometra NEWTONUS *invenit tres ſubſequentes Algebraicarum Curvarum Proprietates*; 1. *Ducantur duæ rectæ inter ſe parallelæ & Curvam ſecantes in tot Punctis, quot ſunt ejus Dimenſiones, & recta, quæ ita ſecat has parallelas, ut Summa partium, inter Curvam & rectam ſecantem jacentium, utrinque ſit inter ſe æqualis, eodem modo ſecabit omnes alias hiſce parallelas.* 2. *Si Curva tot habet Aſymptotos, quot ſunt ejus Dimenſiones, tum Summa partium ordinatæ inter ſingulum crus & ejus aſymptoton interceptarum, utrinque Diametri erit inter ſe æqualis.* 3. *Si Abſciſſa tot habet Vertices, quot ſunt ejus Dimenſiones, tum Contentum ſub omnibus Abſciſſis inter ordinatam & vertices interceptis erit in data ratione ad contentum ſub omnibus ordinatis: enumeravit etiam Lineas tertii Ordinis, & dicit, quod Curvæ* (n) *Dimenſionum projiciunt umbras, quarum Sectiones ſunt Curvæ* (n) *Dimenſionum; & rectificavit Epicycloidem, cujus Aream invenit magnus ille Philoſophus* HALLEIUS.

Celeberrimus COTESIUS *invenit generalem Algebraicarum Linearum Proprietatem, quam illo Viro præſtantiſſimo* MAC LAURIN *communicavit Vir Reverendus* ROB. SMITH, *S.T.P. & Sanctæ Trin. Coll. Cantab. Præfectus, ſummus ille Philoſophus, & in Mathematicis Diſciplinis maxime inſignis. Plures Proprietates exinde deduxit* MAC LAURIN: *ille etiam tractavit organicam Curvarum Deſcriptionem, de quâ prius ſcripſit* NEWTONUS.

Horum autem Laudibus, aliorumque, qui Studiis ſuis haſce Diſciplinas promoverunt, non eſt, ut hìc diutiùs immoremur.

Reſtat, ut de iis rebus pauca dicam, quas complectitur hujus Operis Pars Secunda.

In primo Capite agitur de Algebraicarum Curvarum Proprietatibus, de Diametris, Centris, centrali Curvâ, Summâ Subnormalium, & Subtangentium, &c.

In Secundo de Curvoidibus, ſive Curvis ex Rotatione unius Curvæ ſuper alteram generatis.

In Tertio, de Umbris, quas projiciunt vel Solida vel Curvæ in Solida vel Plana.

In Quarto, tractatur de Fluxionalium Curvarum, & Ovalium in ſe Concavarum Proprietatibus.

In

In Quinto de Enumeratione Algebraicarum Linearum Quarti Ordinis.

Nihil reſtat, niſi ut iis agam Gratias, quorum Nominibus hoc opus præcipuè ornatur; eoſque enixè rogem, ut quâ Humanitate Induſtriam meam adjuvârunt, eâdem Fructum ejus, qualiſcunque ſit, tandem accipiant.

Nec tamen ſcribendi Finem prius faciam quam Gratias maximas agam egregio Adoleſcenti, in omni Scientiarum Genere maximè acuto & verſato, JOANNI WILSON, *Div. Pet. Coll. qui multa & præclara à ſe inventa & excogitata habet, cujus cùm Curæ, tum Ingenio hoc Opuſculum quamplurimum debere lubentiſſimè agnoſco.*

Coll. Magd. Cantab.
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MIS-

MISCELLANEA ANALYTICA.

CAPUT I.

LEMMA I.

SINT $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. *radices datæ æquationis*
$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - vx^{n-5} + wx^{n-6}$. *&c.* $= 0$.
Invenire ſummam $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m$, *&c.*

Summa quæſita erit

$$p^m - mqp^{m-2} + mrp^{m-3} - msp^{m-4}$$
$$+ m.\overline{\frac{m-3}{2}}\, q^2$$

$$\begin{array}{llllll}
+ & mtp^{m-5} & - & mvp^{m-6} & + & mwp^{m-7} \\
- & m.\overline{m-4}\,qr & + & m.\overline{m-5}\,qs & - & m\times\overline{m-6}\times qt \\
 & & - & m\times\overline{\frac{m-5}{2}}\times\overline{\frac{m-4}{3}}\times q^3 & + & m\times\overline{m-5}\times\overline{\frac{m-6}{2}}\times q^2 r \\
 & & + & m.\overline{\frac{m-5}{2}}\, r^2 & - & m\times\overline{m-6}\times rs
\end{array}$$

Duplex eſt lex, quam obſervat hæc ſeries, altera, quæ producta literalia, altera, quæ coefficientes ſeu uncias reſpicit.

Productorum literalium lex eſt, ut datæ æquationis coefficientium exponentes, in ſingulis productis eidem poteſtati quantitatis p annectendis, eandem conficiunt ſummam, quæ quidem ſumma eſt u, ſi modo p^{m-u} ſit poteſtas quantitatis p, cui annectuntur producta hæc literalia.

Per datæ æquationis coefficientium exponentes intelligo numeros deſignantes earum diſtantias a primo æquationis termino; ſic exponens coefficientis p, eſt 1; coefficientis q, 2; coefficientis r, 3; & ſic deinceps; quo fit, ut exponens cujuſlibet poteſtatis coefficientis p, qualis p^u ſit u; exponens cujuſlibet poteſtatis coefficientis q qualis q^{μ}, ſit 2μ; exponens cujuſlibet poteſtatis coefficientis r, qualis r^{ν}, ſit 3ν.

Coefficientium ſeu unciarum hæc eſt lex, cujuſcunque literalis producti $p^{m-u}\, q^{\mu}\, r^{\nu}\, s^{\pi}\, t^{\varrho}$, &c. coefficiens erit
$m\times\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\times\overline{m-u+4}\times\overline{m-u+5}$, &c.
tot factores erunt, quot unitates in ſumma indicum μ, ν, π, ϱ, &c. Dividatur hoc productum per productum ortum ex multiplicatione

ſerierum $1\times2\times3\times4$, &c. $1\times2\times3\times4$, &c. $1\times2\times3\times4$, &c. quarum prima continet tot terminos, quot ſunt unitates in μ, ſecunda, quot ſunt in ν, tertia, quot ſunt in π, quarta, quot ſunt in ϱ, &c.

DEMONSTRATIO.

" Sit $p=a, p^2-2q=b, pb-qa+3r=c, pc-qb+ra-4s=d,$ " $pd-qc+rb-sa+5t=e, pe-qd+rc-sb+ta-6v=f,$ & " ſic deinceps. Et erit a ſumma radicum, b ſumma quadratorum ex " ſingulis radicibus, c ſumma cuborum, d ſumma quadrato-quadra- " torum, e ſumma quadrato-cuborum, f ſumma cubo-cuborum, & " ſic in reliquis." *

Et hâc *Newtoni* regulâ ſatis facile colligitur, coefficientem ſeu unciam literalis producti $p^{m-u}q^{\mu}r^{\nu}s^{\pi}t^{\varrho}$, &c. in ſummâ $\alpha^m+\beta^m+\gamma^m+\delta^m+\varepsilon^m$, &c. inveniendâ, æqualem eſſe aggregato coefficientium ſeu unciarum literalium productorum $p^{m-u-1}q^{\mu}r^{\nu}s^{\pi}t^{\rho}$, &c. $p^{m-u}q^{\mu-1}r^{\nu}s^{\pi}t^{\rho}$ &c. $p^{m-u}q^{\mu}r^{\nu-1}s^{\pi}t^{\rho}$ &c. $p^{m-u}q^{\mu}r^{\nu}s^{\pi-1}t^{\rho}$ &c. $p^{m-u}q^{\mu}r^{\nu}s^{\pi}t^{\rho-1}$, &c. &c. in ſummis reſpectivis $\alpha^{m-1}+\beta^{m-1}+\gamma^{m-1}+\delta^{m-1}$, &c. $\alpha^{m-2}+\beta^{m-2}+\gamma^{m-2}+\delta^{m-2}$, &c. $\alpha^{m-3}+\beta^{m-3}+\gamma^{m-3}+\delta^{m-3}$, &c. $\alpha^{m-4}+\beta^{m-4}+\gamma^{m-4}+\delta^{m-4}$, &c. $\alpha^{m-5}+\beta^{m-5}+\gamma^{m-5}+\delta^{m-5}$, &c.

Hæ autem unciæ vel coefficientes reſpective ſunt

$$\overline{m-1}\times\overline{m-u}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

$$\mu\times\overline{m-2}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

$$\nu\times\overline{m-3}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

$$\pi\times\overline{m-4}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.2.3\ldots\rho,\ \&c.}$$

$$\rho\times\overline{m-5}\times\frac{\overline{m-u+1}\times\overline{m-u-2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

&c.

Aggregatum harum unciarum eſt

$$\left.\begin{array}{lll} m^2-u & \times m+ & u \\ -1 & & -2\mu \\ +\mu & & -3\nu \\ +\nu & & -4\pi \\ +\pi & & -5\rho \\ +\rho & & \&c. \\ \&c. & & \end{array}\right\}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}.}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

In

* *Newt.* Arith. Univ. pag. 251.

In primo hujus aggregati factore, pro u ſubſtituatur ejus valor $(2\mu+3\nu+4\pi+5\rho$, &c.) & fit aggregatum

$$m\times\overline{m-u+\mu+\nu+\pi+\rho,\&c.-1}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\,..\,\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3..\mu.1.2.3..\nu.1.2.3..\pi.1.2.3...\rho,\ \&c.}$$

Hæc vero eſt coefficiens ſeu uncia, quæ in datâ ſerie pro uncia literalis producti $p^{m-u}q^{\mu}r^{\nu}s^{\pi}t^{\rho}$, &c. in ſummâ $\alpha^m+\beta^m+\gamma^m+\delta^m+\varepsilon^m$, &c. inveniendâ, ſubſtituitur; & conſequenter ſi ſeries inferiores ſummas $\alpha^{m-1}+\beta^{m-1}+\gamma^{m-1}+\delta^{m-1}$, &c. $\alpha^{m-2}+\beta^{m-2}+\gamma^{m-2}+\delta^{m-2}$, &c. $\alpha^{m-3}+\beta^{m-3}+\gamma^{m-3}+\delta^{m-3}$, &c. $\alpha^{m-4}+\beta^{m-4}+\gamma^{m-4}+\delta^{m-4}$, &c. recte deducit; patet ex iis, quæ jam dicta fuerunt, proxime ſuperiorem ſummam $\alpha^m+\beta^m+\gamma^m+\delta^m+\varepsilon^m$, &c. etiam ex illâ recte aſſignatam eſſe.

EXEMPLUM.

Sit æquatio $x^4-2x^3-7x^2+20x-12=0$. In hoc exemplo $p,q,r,s,$ ſunt reſpective $2,-7,-20,-12$. Quare ſumma quadratorum ex ſingulis radicibus (p^2-2q) eſt $2^2+2\times7=18$; cuborum ſumma $(p^3-3pq+3r)$ $2^3+3\times2\times7-3\times20=-10$; quadratoqnadratorum ſumma

$$\left(\begin{array}{l}p^4-4p^2q+4pr-4s\\ \qquad\qquad\quad+2q^2\end{array}\right)2^4+4\times2^2\times7-4\times2\times20+4\times12+2\times7^2=114$$

& ſic deinceps.

COROLLARIUM I.

Sit a ſumma radicum, b ſumma quadratorum ex ſingulis radicibus, c ſumma cuborum, d ſumma quadrato-quadratorum, e ſumma quadrato-cuborum, &c.

Invenire ſummam (m) terminorum ab initio hujuſce ſeriei $a+b+c+d+e$, &c.

Summa quæſita eſt

$$\frac{p-p^{m+1}}{1-p}-q\times\frac{2-p-\overline{m+1}p^{m-1}+mp^m}{\overline{1-p}^2}+r\times\frac{3-2p-\overline{m+1}.p^{m-2}+mp^{m-1}}{\overline{1-p}^2}$$

$$-s\times\frac{4-3p-\overline{m+1}p^{m-3}+mp^{m-2}}{\overline{1-p}^2}$$

$$+q^2\times\frac{4-2p-\overline{m+1}\times\overline{m-2}p^{m-3}+\overline{2m^2-4m-4}p^{m-2}-m.\overline{m-3}.p^{m-1}}{2.\overline{1-p}^3}$$

$$+t\times\frac{5-4p-\overline{m+1}.p^{m-4}+mp^{m-3}}{\overline{1-p}^{2}}$$

$$-qr\times\frac{5-3p-\overline{m+1}\times\overline{m-3}.p^{m-4}+\overline{2m^2-6m-5}.p^{m-3}-m.\overline{m-4}.p^{m-2}}{\overline{1-p}^{3}},\text{\&c.}$$

Literalia producta hujuſce ſeriei eadem fiunt, ac literalia producta præcedentis ſeriei, ſi modo abjiciatur quantitas (p) & ejus poteſtates.

Regula autem, quam obſervant coefficientes ſeu unciæ, ſic poterit enunciari.

Scribatur $u=2\mu+3\nu+4\pi+5\rho$, &c. & $l=1+\mu+\nu+\pi+\rho$, &c. Coefficiens literalis producti $q^{\mu}r^{\nu}s^{\pi}t^{\rho}$, &c. erit æqualis producto $1.2.3.4..\overline{l-2}$ in fractionem multiplicato, cujus denominator eſt $\overline{1-p}^{l}\times1\times2\times3..\mu\times1\times2\times3..\nu\times1\times2\times3\times4..\pi\times1\times2\times3..\rho$, &c.

Numerator vero $u-\overline{u-l+1}\times p$

$$\left.\begin{array}{l}-u\times\frac{m-u+2}{1}\times\frac{m-u+3}{2}\times\frac{m-u+4}{3}\times\frac{m-u+5}{4}\times\frac{m-u+6}{5}\dots\frac{m-u+l}{l-1}\\ +\overline{u-l+1}\times\frac{m-u+1}{1}\times\frac{m-u+2}{2}\times\frac{m-u+3}{3}\times\frac{m-u+4}{4}\times\frac{m-u+5}{5}\dots\frac{m-u+l-1}{l-1}\end{array}\right\}p^{m-u+1}$$

$$\left.\begin{array}{l}+u\times\frac{m-u+1}{1}\times\frac{m-u+3}{1}\times\frac{m-u+4}{2}\times\frac{m-u+5}{3}\times\frac{m-u+6}{4}\dots\frac{m-u+l}{l-2}\\ -\overline{u-l+1}\times\frac{m-u}{1}\times\frac{m-u+2}{1}\times\frac{m-u+3}{2}\times\frac{m-u+4}{3}\times\frac{m-u+5}{3}\dots\frac{m-u+l-1}{l-2}\end{array}\right\}p^{m-u+2}$$

$$\left.\begin{array}{l}-u\times\frac{m-u+1}{1}\times\frac{m-u+2}{2}\times\frac{m-u+4}{1}\times\frac{m-u+5}{2}\times\frac{m-u+6}{3}\dots\frac{m-u+l}{l-3}\\ +\overline{u-l+1}\times\frac{m-u}{1}\times\frac{m-u+1}{2}\times\frac{m-u+3}{1}\times\frac{m-n+4}{2}\times\frac{m-u+5}{3}\dots\frac{m-u+l-1}{l-3}\end{array}\right\}p^{m-u+3}$$

$$\left.\begin{array}{l}+u\times\frac{m-u+1}{1}\times\frac{m-u+2}{2}\times\frac{m-u+3}{3}\times\frac{m-u+5}{1}\times\frac{m-u+6}{2}\dots\frac{m-u+l}{l-4}\\ -\overline{u-l+1}\times\frac{m-u}{1}\times\frac{m-u+1}{2}\times\frac{m-u+2}{3}\times\frac{m-u+4}{1}\times\frac{m-u+5}{2}\dots\frac{m-u+l-1}{l-4}\end{array}\right\}p^{m-u+4}$$

$$\left.\begin{array}{l}-u\times\frac{m-u+1}{1}\times\frac{m-u+2}{2}\times\frac{m-u+3}{3}\times\frac{m-u+4}{4}\times\frac{m-u+6}{1}\dots\frac{m-u+l}{l-5}\\ +\overline{u-l+1}\times\frac{m-u}{1}\times\frac{m-u+1}{2}\times\frac{m-u+2}{3}\times\frac{m-u+3}{4}\times\frac{m-u+5}{1}\dots\frac{m-u+l-1}{l-5}\end{array}\right\}p^{m-u+5}$$

. . . .

$$\left.\begin{array}{l}+m\times\frac{m-u+1}{1}\times\frac{m-u+2}{2}\times\frac{m-u+3}{3}\times\frac{m-u+4}{4}\times\frac{m-u+5}{5}\dots\frac{m-u+l-1}{l-1}\\ +\overline{u-l+1}\times\frac{m-u}{1}\times\frac{m-u+1}{2}\times\frac{m-u+2}{3}\times\frac{m-u+3}{4}\times\frac{m-u+4}{5}\dots\times\frac{m-u+l-2}{l-1}\end{array}\right\}p^{m-u+l}$$

Ex methodo inveniendi ſummam $(m-u+1)$ terminorum ſeriei emergentis ex diviſione quantitatis $u-\overline{u-l+1}\times p$ per $\overline{1-p}^{\,l}$, facile erui & demonſtrari poteſt hoc corollarium.

EXEMPLUM.

Datâ æquatione $x^4-3x^3+x^2+3x-2=0$, invenire ſummam $a+b+c+d$.

Pro p, q, r, s, m ſcribantur reſpective $3, 1, -3, -2, 4$.

$$a+b+c+d=\frac{p-p^5}{1-p}(120)-q\times\frac{2-p-5p^3+4p^4}{\overline{1-p}^2}(-47)$$

$$+r\times\frac{3-2p-5p^2+4p^3}{\overline{1-p}^2}(-45)-s\times\frac{4-3p-5p+4p^2}{\overline{1-p}^2}(8)=38.$$

$$+q^2\times\frac{4-2p-10p+12p^2-4p^3}{2.\overline{1-p}^3}(2)$$

Si $p=1$, tum $\frac{p-p^{m+1}}{1-p}$ fit (m) & $q\times\frac{2-p-\overline{m+1}.p^{m-1}+mp^m}{\overline{1-p}^2}$ fit $\left(q\times\overline{m+2}\times\frac{\overline{m-1}}{2}\right)$. &c.

Summa quæſita eſt

$$m-\frac{\overline{m+2}}{1}\times\frac{\overline{m-1}}{2}q+\frac{\overline{m+3}}{1}\times\frac{\overline{m-2}}{2}r-\frac{\overline{m+4}}{1}\times\frac{\overline{m-3}}{2}\times s,\ \&c.$$

$$+\tfrac{1}{2}\times\frac{\overline{2m+4}}{1}\times\frac{\overline{m-3}}{2}\times\frac{\overline{m-2}}{3}q^2,\ \&c.$$

Coefficientium ſeu unciarum hujuſce ſeriei hæc eſt lex. Coefficiens cujuſlibet literalis producti $q^\mu r^\nu s^\pi t^\rho$, &c. æqualis eſt fractioni,

$$\overline{\overline{l-1}\times m+u}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\ldots\overline{m-u+l-1},\ \&c.}{l\times\overline{l-1}\times1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho,\ \&c.}$$

COROLLARIUM II.

Invenire ſummam ſeriei $a+b+c+d+e$, &c. in infinitum.

Summa quæſita eſt $\frac{p-2q+3r-4s+5t,\ \&c.}{1-p+q-r+s-t,\ \&c.}$

EX-

EXEMPLUM.

Sit æquatio $x^4 - x^3 + \frac{1}{4}x - \frac{1}{16} = 0$, ſcribatur $1, 0, -\frac{1}{4}, -\frac{1}{16}$ pro p, q, r, s, & $\frac{p - 2q + 3r - 4s}{1 - p + q - r + s}$ fit $\frac{1 - \frac{3}{4} + \frac{1}{4}}{1 - 1 + \frac{1}{4} - \frac{1}{16}} = \frac{\frac{1}{2}}{\frac{3}{16}} = 2\frac{2}{3}$.

Quam plurima hujuſce generis problemata deduci poſſunt, datam æquationem in alteram transformando, cujus radices vel ſunt reciprocæ vel quadraticæ, cubicæ, &c. radices ex quibuſlibet complexis functionibus datæ æquationis radicum ($\alpha, \beta, \gamma, \delta$, &c.)

LEMMA II.

Sint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. *radices datæ æquationis* $x^n - px^{n-1} + q^{n-2} - rx^{n-3}$ &c. $= 0$ *& ſint* a, b, c, d, e, f, &c. *dati indices, cujus numerus ſit* (m), & m *vel æqualis vel minor ſit quam* n,

Invenire ſummam omnium quantitatum hujuſmodi.
$\alpha^a \beta^b \gamma^c \delta^d \varepsilon^e$ &c. $+ \alpha^b \beta^a \gamma^c \delta^d \varepsilon^e$ &c. in quibus numerus factorum eſt (m).

Scribatur

$S^a = - \alpha^a + \beta^a + \gamma^a + \delta^a + \varepsilon^a$, &c.
$S^b = - \alpha^b + \beta^b + \gamma^b + \delta^b + \varepsilon^b$, &c.
$S^c = - \alpha^c + \beta^c + \gamma^c + \delta^c + \varepsilon^c$, &c.
$S^d = - \alpha^d + \beta^d + \gamma^d + \delta^d + \varepsilon^d$, &c.

$S^{a+b} = - - - - - - - - - - - - - - - - \alpha^{a+b} + \beta^{a+b} + \gamma^{a+b} + \delta^{a+b}$, &c.
$S^{a+c} = - - - - - - - - - - - - - - - - \alpha^{a+c} + \beta^{a+c} + \gamma^{a+c} + \delta^{a+c}$, &c.
$S^{a+d} = - - - - - - - - - - - - - - - - \alpha^{a+d} + \beta^{a+d} + \gamma^{a+d} + \delta^{a+d}$, &c.

$S^{a+b+c} = - - - - - - - - - - - - \alpha^{a+b+c} + \beta^{a+b+c} + \gamma^{a+b+c} + \delta^{a+b+c}$, &c.
$S^{a+b+d} = - - - - - - - - - - - - \alpha^{a+b+d} + \beta^{a+b+d} + \gamma^{a+b+d} + \delta^{a+b+d}$, &c.
$S^{a+c+d} = - - - - - - - - - - - - \alpha^{a+c+d} + \beta^{a+c+d} + \gamma^{a+c+d} + \delta^{a+c+d}$, &c.

$S^{a+b+c+d} = - - - - - - - - - - \alpha^{a+b+c+d} + \beta^{a+b+c+d} + \gamma^{a+b+c+d}$, &c.
$S^{a+b+c+e} = - - - - - - - - - - - \alpha^{a+b+c+e} + \beta^{a+b+c+e} + \gamma^{a+b+c+e}$, &c.

$S^{a+b+c+d+e} = - - - - - - \alpha^{a+b+c+d+e} + \beta^{a+b+c+d+e} + \gamma^{a+b+c+d+e}$, &c.
$S^{a+b+c+d+f} = - - - - - - \alpha^{a+b+c+d+f} + \beta^{a+b+c+d+f} + \gamma^{a+b+c+d+f}$, &c.

& ſic deinceps.

Scribatur

$$A = S^a \times S^b \times S^c \times S^d \times S^e, \text{ \&c.}$$

$$B = \frac{S^{a+b} \times A}{S^a \times S^b} + \frac{S^{a+c} \times A}{S^a \times S^c} + \frac{S^{a+d} \times A}{S^a \times S^d}, \text{ \&c.}$$

$$C = \frac{S^{a+b+c} \times A}{S^a \times S^b \times S^c} + \frac{S^{a+b+d} \times A}{S^a \times S^b \times S^d} + \frac{S^{a+c+d} \times A}{S^a \times S^c \times S^d}, \text{ \&c.}$$

$$D = \frac{S^{a+b+c+d} \times A}{S^a \times S^b \times S^c \times S^d} + \frac{S^{a+b+c+e} \times A}{S^a \times S^b \times S^c \times S^e}, \text{ \&c.}$$

$$BB = \frac{S^{a+b} \times S^{c+d} \times A}{S^a \times S^b \times S^c \times S^d} + \frac{S^{a+c} \times S^{b+d} \times A}{S^a \times S^b \times S^c \times S^d} + \frac{S^{a+e} \times S^{b+c} \times A}{S^a \times S^e \times S^b \times S^c}, \text{ \&c.}$$

$$E = \frac{S^{a+b+c+d+e} \times A}{S^a \times S^b \times S^c \times S^d \times S^e} + \frac{S^{a+b+c+d+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^f}, \text{ \&c.}$$

$$BC = \frac{S^{a+b} \times S^{c+d+e} \times A}{S^a \times S^b \times S^c \times S^d \times S^e} + \frac{S^{a+c} \times S^{d+e+f} \times A}{S^a \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

$$F = \frac{S^{a+b+c+d+e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

$$BD = \frac{S^{a+b+c+d} \times S^{e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f} + \frac{S^{a+b+c+e} \times S^{d+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^f}, \text{ \&c.}$$

$$CC = \frac{S^{a+b+c} \times S^{d+e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f} + \frac{S^{a+b+d} \times S^{c+e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

$$BBB = \frac{S^{a+b} \times S^{c+d} \times S^{e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f} + \frac{S^{a+c} \times S^{b+d} \times S^{e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

& ſic deinceps.

Hic obſervandum eſt, quod per S^a, S^b, S^c, &c. S^{a+b}, &c. non intelligo poteſtates (a, b, c, &c. $a+b$, &c.) quantitatis S, eodem modo per BB, BC, &c. non intelligo rectangulum ſub B & B vel C contentum, ſed notationem deſignantem quantitatem, quæ ſibi ipſi per notam æqualitatis adjungitur.

Summa quæſita eſt

$$A - B + 1 \times 2\,C - 1 \times 2 \times 3\,D + 1 \times 2 \times 3 \times 4\,E$$
$$+ 1 \times 1\,BB - 1 \times 1 \times 2\,BC$$

$$- 1 \times 2 \times 3 \times 4 \times 5\,F \quad + 1 \times 2 \times 3 \times 4 \times 5 \times 6\,G$$
$$+ 1 \times 1 \times 2 \times 3 \times BD \quad - 1 \times 1 \times 2 \times 3 \times 4\,BE$$
$$- 1 \times 1 \times 1\,BBB \quad + 1 \times 1 \times 1 \times 2 \times BBC$$
$$+ 1 \times 2 \times 1 \times 2\,CC \quad - 1 \times 2 \times 1 \times 2 \times 3\,CD$$

Literalium quantitatum hæc eſt lex.

Exponentes literalium quantitatum terminorum infra ſe poſitorum eandem conficiunt ſummam, quæ quidem ſumma erit ſucceſſive 1, 2, 3, 4, 5, 6, &c.

Per exponentes literarum intelligo numeros deſignantes ſedes illas, quas in ſerie naturali literarum ſingulæ literæ occupant, ſic exponens literæ *A* eſt 1; literæ *B*, 2; literæ *C*, 3; & ſic deinceps, quo fit ut exponens literalis quantitatis *BBB*...*CCC*...*DDD*...*EEE*, &c. (in quâ ſint tot literæ *B*, *C*, *D*, *E*, &c. contentæ, quot unitates in π, ρ, σ, τ, &c. reſpective) ſit $2\pi + 3\rho + 4\sigma + 5\tau$, &c.

Coefficiens literalis quantitatis *BBB*...*CCC*...*DDD*...*EEE*, &c. in quâ tot ſunt literæ *B*, quot unitates in π, tot ſunt literæ *C*, quot unitates in ρ, tot ſunt literæ *D*, *E*, &c. quot unitates in σ, τ, &c. reſpective, æqualis eſt producto $1^\pi \times 1^\rho \times 2^\rho \times 1^\sigma \times 2^\sigma \times 3^\sigma \times 1_\tau \times 2^\tau \times 3^\tau \times 4^\tau$, &c.

Si duo vel tres vel quatuor vel plures indices (*a*, *b*, *c*, *d*, &c.) ſunt inter ſe æquales, tum dividatur ſumma per producta
1×2, $1 \times 2 \times 3$, $1 \times 2 \times 3 \times 4$, &c. reſpective & ſic conjunctim.

EXEMPLUM I.

Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis,
$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}$, &c. $= 0$. & inveniantur

$$P^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^3, \text{ \&c.} = 16.$$
$$P^3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3, \text{ \&c.} = 30.$$
$$\overline{P^{2+3}} = \alpha^5 + \beta^5 + \gamma^5 + \delta^5, \text{ \&c.} = 265.$$

invenire ſummam,

$$\alpha^2 \times \beta^3 + \alpha^2 \times \gamma^3 + \alpha^2 \times \delta^3 + \text{ \&c.} + \beta^2 \times \alpha^3 + \beta^2 \times \gamma^3 + \beta^2 \times \delta^3 + \text{ \&c.}$$
$$+ \gamma^2 \times \alpha^3 + \gamma^2 \times \beta^3 + \gamma^2 \times \delta^3 + \text{ \&c.} + \delta^2 \times \alpha^3 + \delta^2 \times \beta^3 + \delta^2 \times \gamma^3 + \text{ \&c.}$$

In hoc exemplo eſt,

$$A = P^2 \times P^3 = 16 \times 30, \quad B = P^{2+3} = 265.$$

ſumma quæſita

$$(A - B) \text{ eſt } 16 \times 30 - 265 = 215.$$

EX-

EXEMPLUM II.

Sint,

$$P^1 = \alpha + \beta + \gamma + \delta, \text{ \&c.} \qquad = 7$$
$$P^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{ \&c.} \qquad = 12$$
$$P^3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \text{ \&c.} \qquad = 18$$
$$P^{1+2} = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \text{ \&c.} = 18$$
$$P^{1+3} = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \text{ \&c.} = 30$$
$$P^{2+3} = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 \text{ \&c.} \qquad = 45$$
$$P^{1+2+3} = \alpha^6 + \beta^6 + \gamma^6 + \delta^6, \text{ \&c.} = 60.$$

invenire ſummam,

$$\alpha\beta^2\gamma^3 + \alpha\beta^2\delta^3 + \alpha\beta^3\gamma^2 + \alpha\beta^3\delta^2 + \beta\alpha^2\gamma^3 + \beta\alpha^2\delta^3 + \beta\alpha^3\gamma^2 + \beta\alpha^3\delta^2 + \text{ \&c.}$$

Conſtat

$$A = P^1 \times P^2 \times P^3 = 7 \times 12 \times 18 = 1512$$
$$B = P^{1+2} \times P^3 + P^{1+3} \times P^2 + P^{2+3} \times P^1 = \overline{18}^2 + 30 \times 12 + 45 \times 7 = 999$$
$$C = P^{1+2+3} = 60.$$

Summa quæſita eſt

$$A - B + 2C = 1512 - 999 + 2 \times 60 = 633.$$

COROLLARIUM I.

Sit (a) ſumma radicum, (b) ſumma quadratorum ex ſingulis radicibus, (c) ſumma cuborum, (d) ſumma quadrato-quadratorum, (e) ſumma quadrato-cuborum, f ſumma cubo-cuborum, &c.

Summa contentorum ſub ſingulis (m) radicibus erit æqualis fractioni, cujus numerator eſt

$$a^m - m \times \frac{\overline{m-1}}{2} \times a^{m-2}b + m \times \overline{m-1} \times \frac{\overline{m-2}}{3} \times a^{m-3}c - m \times \overline{m-1} \times \overline{m-2} \times \frac{\overline{m-3}}{4} \times a^{m-4}d \text{ \&c.}$$
$$+ m \times \overline{m-1} \times \frac{\overline{m-2}}{2} \times \frac{m-3}{2^2}\, a^{m-4}b^2$$

Denominator vero $1 \times 2 \times 3 \times 4 \ldots \times m$.

Scribantur pro literis a, b, c, d, e, &c. p, q, r, s, t, &c. reſpective; & hujus ſeriei literalia producta eadem erunt, ac literalia producta ſeriei in lemmate primo.

Coefficiens literalis producti $(a^{m-l} \times b^{\pi} \times c^{\rho} \times d^{\sigma}$, &c.) erit æqualis fractioni, cujus numerator eſt $m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}$, &c. tot factores erunt, quot unitates in ſummâ $2\pi + 3\rho + 4\sigma$, &c. Denominator vero æqualis erit producto orto ex continuâ multiplicatione ſerierum $1 \times 2 \times 3$, &c. $1 \times 2 \times 3$, &c. $1 \times 2 \times 3$, &c. $2^{\pi} 3^{\rho} 4^{\sigma}$, &c. quarum prima continet tot terminos, quot unitates in π, ſecunda, quot ſunt unitates in ρ, tertia, quot ſunt in σ, & ſic deinceps.

COROLLARIUM II.

Sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. radices datæ æquationis,

$x^n - px^{n-1} + qx^{n-2} - rx^{n-3}$, &c. $= 0$. & ſit fractio $\dfrac{a - bx + cx^2 - dx^3, \&c.}{A - Bx + Cx^2 - Dx^3, \&c.}$ in quâ ſint coefficientes (a, b, c, &c. A, B, C, &c.) datæ,

Invenire ſummam fractionum,

$$\frac{a - b\alpha + c\alpha^2 - d\alpha^3, \&c.}{A - B\alpha + C\alpha^2 - D\alpha^3, \&c.} + \frac{a - b\beta + c\beta^2 - d\beta^3, \&c.}{A - B\beta + C\beta^2 - D\beta^3, \&c.} + \frac{a - b\gamma + c\gamma^2 - d\gamma^3, \&c.}{A - B\gamma + C\gamma^2 - D\gamma^3, \&c.} \ \&c.$$

Reducantur omnes hæ fractiones in communem denominatorem, tum lemmatis primi & ſecundi ope inveniri poteſt & numerator & denominator.

EXEMPLUM.

Sit data æquatio $x^3 - 7x + 6 = 0$, cujus radices ſint α, β, γ, & fractio $\dfrac{3x^2+5}{4x+1}$.

Invenire ſummam fractionum,

$\dfrac{3\alpha^2+5}{4\alpha+1}, \dfrac{3\beta^2+5}{4\beta+1}, \dfrac{3\gamma^2+5}{4\gamma+1}$, reducantur hæ fractiones in communem denominatorem & fiunt,

$$\frac{48 \times \alpha\beta\gamma \times \overline{\alpha+\beta+\gamma} + 12 \times \overline{\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta}}{64 \times \alpha\beta\gamma + 16 \times \overline{\alpha\beta + \alpha\gamma + \beta\gamma} + 4 \times \overline{\alpha+\beta+\gamma} + 1} +$$

$$\frac{3 \times \overline{\alpha^2+\beta^2+\gamma^2} + 80 \times \overline{\alpha\beta + \alpha\gamma + \beta\gamma} + 40 \times \overline{\alpha+\beta+\gamma} + 15}{64 \times \alpha\beta\gamma + 16 \times \overline{\alpha\beta + \alpha\gamma + \beta\gamma} + 4 \times \overline{\alpha+\beta+\gamma} + 1}$$

Pro p, q, r, ſcribantur 0, -7, & -6 tum

De-

Denominator eſt

$$64\times\alpha\beta\gamma(64\times r=64\times-6)+16\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}(16\times q=16\times-7)+$$
$$4\times\overline{\alpha+\beta+\gamma}\,(4\times p=0)+1=-495.$$

Numerator vero

$$48\times\alpha\beta\gamma\times\overline{\alpha+\beta+\gamma}\,(48\,rp=0)+$$
$$12\times\overline{\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta}\,(12\times\overline{pq-3r}=12\times3\times6)+$$
$$3\times\overline{\alpha^2+\beta^2+\gamma^2}\,(3\times\overline{p^2-2q}=3\times2\times7)+80\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}$$
$$(80\times q=80\times-7)+40\times\overline{\alpha+\beta+\gamma}\,(40\times p=40\times0)+15=-287.$$

& conſequenter ſumma quæſita eſt $\frac{287}{495}$.

Eodem modo inveniri poteſt aggregatum irrationalium functionum datæ æquationis radicum.

PROB. I.

Datâ unâ, duabus, vel pluribus æquationibus, invenire æquationem, cujus radices, ad radices datarum æquationum quamcunque datam relationem (exponentialibus exceptis) habent.

Scribantur $\alpha, \beta, \gamma, \delta$, &c. pro radicibus unius datæ æquationis, A, B, Γ, Δ, &c. pro radicibus alterius datæ æquationis & ſic de reliquis.

Ex datâ aſſignabili relatione inveniantur omnes quæſitæ æquationis radices, & lemmatis primi vel ſecundi ope inveniri poteſt ex ſingulis hiſce radicibus aggregatum, quod eſt coefficiens ſecundi quæſitæ æquationis termini.

Ducantur quæque duæ radices quæſitæ æquationis in ſeſe & ex lemmatibus prædictis inveniatur aggregatum rectangulorum reſultantium & erit coefficiens tertii quæſitæ æquationis termini.

Ducantur quæque tres, quatuor, quinque, &c. radices quæſitæ æquationis continuo in ſeſe & ex iiſdem lemmatibus inveniantur aggregata reſultantium productorum & hæc aggregata erunt reſpective coefficientes quarti, quinti, ſexti, &c. quæſitæ æquationis terminorum.

Hinc dilucide conſtat methodus inveniendi quæſitæ æquationis coefficientes, & conſequenter ipſam æquationem.

Hujuſce generis problemata ſæpe ſolvi poſſunt ex reductione duarum vel plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

EXEMPLUM I.

Sint $\alpha, \beta, \gamma, \delta$ &c. Radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}, \text{ \&c.} = 0.$$

invenire æquationem

$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$, &c. $= 0$. cujus radices ſunt ($\alpha^m, \beta^m, \gamma^m, \delta^m$, &c, reſpective) m poteſtates radicum datæ æquationis.

Scribatur

$$a = \alpha^m + \beta^m + \gamma^m + \delta^m, \text{ \&c.}$$
$$b = \alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m}, \text{ \&c.}$$
$$c = \alpha^{3m} + \beta^{3m} + \gamma^{3m} + \delta^{3m}, \text{ \&c.}$$
$$d = \alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m}, \text{ \&c.}$$

Tum ex primo lemmatis ſecundi corollario inveniantur requiſitæ æquationis coefficientes (P, Q, R, S, &c.) & erunt reſpective

$$P = \alpha^m + \beta^m + \gamma^m + \delta^m, \text{ \&c.} = a$$

$$Q = \alpha^m\times\beta^m + \alpha^m\times\gamma^m + \alpha^m\times\delta^m + \text{\&c.} + \beta^m\times\gamma^m + \beta^m\times\delta^m + \text{\&c.} + \gamma^m\times\delta^m + \text{\&c.} = \frac{a^2 - b}{2}$$

$$R = \alpha^m\times\beta^m\times\gamma^m + \alpha^m\times\beta^m\times\delta^m + \alpha^m\times\gamma^m\times\delta^m + \text{\&c.} + \beta^m\times\gamma^m\times\delta^m + \text{\&c.} = \frac{a^3 - 3ab + 2c}{6}$$

$$S = \alpha^m\times\beta^m\times\gamma^m\times\delta^m + \text{ \&c.} = \frac{a^4 - 6a^2b + 8ac - 6d + 3b^2}{24}$$

& ſic de reliquis coefficientibus.

Ex lemmate ſecundo ab aliâ methodo derivatur hujus exempli ſubſequens ſolutio.

Sint h, k, l, o, &c. omnes diverſæ radices m poteſtatis unitatis.

Tum ex hypotheſi $x^m = v$ & conſequenter

$$hv^{\frac{1}{m}} = x,\ kv^{\frac{1}{m}} = x,\ lv^{\frac{1}{m}} = x,\ ov^{\frac{1}{m}} = x, \text{ \&c.}$$

Quibus valoribus in datâ æquatione pro quantitate (x) ſubſtitutis reſultant æquationes.

$$h^n v^{\frac{n}{m}} - p\times h^{n-1} v^{\frac{n-1}{m}} + q\times h^{n-2} v^{\frac{n-2}{m}} - r\times h^{n-3}\times v^{\frac{n-3}{m}}, \text{ \&c.} = 0.$$

$$k^n v^{\frac{n}{m}} - p\times k^{n-1}\times v^{\frac{n-1}{m}} + q\times k^{n-2} v^{\frac{n-2}{m}} - r\times k^{n-3}\times v^{\frac{n-3}{m}}, \text{ \&c.} = 0.$$

$$l^n v^{\frac{n}{m}} - p\times l^{n-1} v^{\frac{n-1}{m}} + q\times l^{n-2} v^{\frac{n-2}{m}} - rl^{n-3}\times v^{\frac{n-3}{m}}, \text{ \&c.} = 0.$$

$$o^n v^{\frac{n}{m}} - p\times o^{n-1} v^{\frac{n-1}{m}} + q\times o^{n-2} v^{\frac{n-2}{m}} - ro^{n-3}\times v^{\frac{n-3}{m}}, \text{ \&c.} = 0.$$

& ſic deinceps.

Du-

Ducantur hæ æquationes continuo in ſeſe, & æquatio reſultans erit $v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$, &c. $= 0$.
cujus radices ſunt m poteſtates radicum datæ æquationis.

Ex lemmate ſecundo inveſtigari poſſunt coefficientes (P, Q, R, S, &c.) & quo minor eſt numerus (m), eo magis ſimplex eſt lex, quam obſervant quæſitæ æquationis coefficientes.

Ultimus quæſitæ æquationis terminus eſt (m) poteſtas ultimi in datâ æquatione termini.

Ex lemmate primo conſtat etiam ab alia methodo hujus exempli ſolutio.

COROLLARIUM.

Hinc deleri poſſunt quæcunque ſurdæ quantitates ex datâ æquatione.

Sit ſurda quantitas delenda $x^{\frac{1}{n}}$ & data æquatio

$$Ax^{\frac{\overline{n-1}}{n}} - Bx^{\frac{\overline{n-2}}{n}} + Cx^{\frac{\overline{n-3}}{n}}, \text{ \&c.} = Q$$

aſſumatur $v = x^{\frac{1}{n}}$ & invenietur ex exemplo, æquatio, cujus radices ſunt v; & pro v ſubſtituatur x & fit.

EXEMPLUM II.

Invenire æquationem, cujus radices ſunt ($\alpha+\beta, \alpha+\gamma, \alpha+\delta$, &c. $\beta+\gamma, \beta+\delta$, &c. $\gamma+\delta$, &c.) ſummæ ex quibuſcunque duabus datæ æquationis radicibus.

Numerus quæſitæ æquationis radicum & conſequenter ejus dimenſiones erit æqualis fractioni $\frac{n\times n-1}{1\times 2}$.

Summa radicum eſt $\overline{n-1}\times\overline{\alpha+\beta+\gamma+\delta}$, &c. $=\overline{n-1}\times p$ quæ eſt coefficiens ſecundi quæſitæ æquationis termini.

Ducantur quæque duæ radices quæſitæ æquationis in ſeſe & aggregatum reſultantium rectangulorum
($\overline{\alpha+\beta}\times\overline{\alpha+\gamma}$, $\overline{\alpha+\beta}\times\overline{\alpha+\delta}$, $\overline{\alpha+\gamma}\times\overline{\alpha+\delta}$, &c. $\overline{\beta+\gamma}\times\overline{\alpha+\beta}$, $\beta+\gamma\times\alpha+\gamma$, &c. &c.) erit

$$\overline{n-1}\times\frac{\overline{n-2}}{2}\times\overline{\alpha^2+\beta^2+\gamma^2+\delta^2}, \text{ \&c.} + n\times\overline{n-2}\times\overline{\alpha\beta+\alpha\gamma+\alpha\delta}, \text{ \&c.}$$
$$\overline{+\beta\gamma+\beta\delta, \text{ \&c.} + \gamma\delta, \text{ \&c.}}$$
ſed ex lemmate primo conſtat

$\overline{n-1}$

$$\overline{n-1}\times\frac{\overline{n-2}}{2}\,\overline{\alpha^2+\beta^2+\gamma^2+\delta^2}\ \&c. \quad = \overline{n-1}\times\frac{\overline{n-2}}{2}\times\overline{p^2-2q}$$

$$n\times\overline{n-2}\times\overline{\alpha\beta+\alpha\gamma+\alpha\delta+\&c.+\beta\gamma+\beta\delta+\&c.+\gamma\delta+}\ \&c. \quad = n\times\overline{n-2}\times q$$

Aggregatum eſt $\overline{n-1}\times\frac{\overline{n-2}}{2}\times p^2+\overline{n-2}\times q$

Sed hoc aggregatum eſt coefficiens tertii datæ æquationis termini.

Ducantur quæque tres radices quæſitæ æquationis in ſeſe ($\overline{\alpha+\beta}\times\overline{\alpha+\gamma}\times\overline{\alpha+\delta}$, $\overline{\alpha+\beta}\times\overline{\alpha+\gamma}\times\overline{\beta+\delta}$, $\overline{\alpha+\beta}\times\overline{\alpha+\gamma}\times\overline{\gamma+\delta}$, &c.)

Aggregatum horum productorum eſt

$$\overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}\times\overline{\alpha^3+\beta^3+\gamma^3+\delta^3}\ \&c. + \overline{n-2}\times\frac{\overline{n^2-2n-1}}{2}\times\overline{\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\&c.+\beta^2\alpha+\beta^2\gamma+\beta^2\delta+\&c.+\gamma^2\alpha+\gamma^2\beta+\gamma^2\delta+\&c.}$$
$$+\overline{n-1}\times\overline{n^2-2n-2}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+}\ \&c.$$

Sed ex lemmate primo & ſecundo colligitur

$$\overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}\times\overline{\alpha^3+\beta^3+\gamma^3+\delta^3}\ \&c. = \overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}\times\overline{p^3-3pq+3r}$$

$$\overline{n-2}\times\frac{\overline{n^2-2n-1}}{2}\times\overline{\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\&c.+\beta^2\alpha+\beta^2\gamma+\beta^2\delta+\&c.+\gamma^2\alpha+}\ \&c. = \overline{n-2}\times\frac{\overline{n^2-2n-1}}{2}\times\overline{pq-3r}$$

$$\overline{n-1}\times\overline{n^2-2n-2}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.} = \overline{n-1}\times\overline{n^2-2n-2}\times r$$

Aggregatum eſt $= n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times p^3-\overline{n-2}^{\,2}\times pq+\overline{n-4}\times r$

Hoc vero aggregatum eſt coefficiens quarti quæſitæ æquationis termini.

Eodem modo inveniri poſſunt reliquæ coefficientes, & conſequenter ſi modo v ſit radix quæſitæ æquationis æquatio quæſita erit

$$v^{n\times\frac{\overline{n-1}}{2}}-\overline{n-1}\times pv^{n.\frac{\overline{n-1}}{2}-1}+\overline{\overline{n-1}\times\frac{\overline{n-2}}{2}p^2+\overline{n-2}\times q}\times v^{n.\frac{\overline{n-1}}{2}-2}-\overline{n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times p^3+\overline{n-2}^{\,2}\times pq+\overline{n-4}\times r}\,v^{n\times\frac{\overline{n-1}}{2}-3}\ \&c. = 0.$$

EXEMPLUM III.

Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis

$$x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+sx^{n-4}\ \&c. = 0.$$

& A, B, Γ, Δ radices datæ æquationis

$$z^m-Pz^{m-1}+Qz^{m-2}-Rz^{m-3}, \&c. = 0.$$

invenire æquationem, cujus radices ſint (αA, αB, $\alpha\Gamma$, $\alpha\Delta$, &c.

βA,

$\beta A, \beta B, \beta\Gamma, \beta\Delta$, &c. $\gamma A, \gamma B, \gamma\Gamma, \gamma\Delta$, &c. $\delta A, \delta B, \delta\Gamma, \delta\Delta$, &c.) Rectangula ſub ſingulâ unius datæ æquationis radice & ſingulâ alterius radice contenta.

Numerus radicum requiſitæ æquationis & conſequenter ejus dimenſiones æqualis eſt rectangulo ſub indicibus $(n \times m)$ datarum æquationum.

Summa radicum & conſequenter ſecundi termini requiſitæ æquationis coefficiens æqualis eſt rectangulo $\overline{\alpha+\beta+\gamma+\delta}$, &c. $\times$ $\overline{A+B+\Gamma+\Delta}$, &c. $= P \times p$.

Ducantur quæque duæ radices datæ æquationis in ſeſe ($\alpha A \times \alpha B$, $\alpha A \times \alpha\Gamma$, $\alpha A \times \alpha\Delta$, &c. $\beta A \times \alpha B$, $\beta A \times \alpha\Gamma$, $\beta A \times \alpha A$, &c. $\gamma A \times \alpha A$, $\gamma A \times \alpha B$, &c.) & rectangulorum reſultantium aggregatum, ſeu coefficiens tertii quæſitæ æquationis termini eſt

$$\overline{\alpha^2+\beta^2+\gamma^2+\delta^2 \text{ \&c.}} \times \overline{AB+A\Gamma+A\Delta+\text{\&c.}+B\Gamma+B\Delta+\text{\&c. } \Gamma\Delta} +$$
$$\overline{A^2+B^2+\Gamma^2+\Delta^2 \text{ \&c.}} \times \overline{\alpha\beta+\alpha\gamma+\alpha\delta+\text{\&c.}+\beta\gamma+\beta\delta+\text{\&c.}+\gamma\delta} +$$
$$2 \times \overline{\alpha\beta+\alpha\gamma+\alpha\delta+\text{\&c.}+\beta\gamma+\beta\delta+\text{\&c.}+\gamma\delta+\text{\&c.}} \times \overline{AB+A\Gamma+A\Delta+\text{\&c.}+B\Gamma+B\Delta+\text{\&c.}+\Gamma A}$$

Ex lemmate primo colligitur

$$\overline{\alpha^2+\beta^2+\gamma^2+\delta^2+\text{\&c.}} \times \overline{AB+A\Gamma+A\Delta+\text{\&c.}+B\Gamma+B\Delta+\text{\&c.}+\Gamma\Delta \text{\&c.}} = \overline{p^2-2q} \times Q$$
$$\overline{A_2+B^2+\Gamma^2+\Delta^2 \text{\&c.}} \times \overline{\alpha\beta+\alpha\gamma+\alpha\delta+\text{\&c.}+\beta\gamma+\beta\delta+\text{\&c.}+\gamma\delta \text{\&c.}} = \overline{P^2-2Q} \times q$$
$$2 \times \overline{\alpha\beta+\alpha\gamma+\alpha\delta+\text{\&c.}+\beta\gamma+\beta\delta+\text{\&c.}+\gamma\delta+\text{\&c.}} \times \overline{AB+A\Gamma+A\Delta+\text{\&c.}+B\Gamma+B\Delta+\text{\&c.}+\Gamma\Delta+\text{\&c.}} = 2Qq$$

Eorum aggregatum eſt $\overline{p^2Q+P^2q-2Qq}$.

Ex eâdem ratiocinandi methodo coefficiens quarti quæſitæ æquationis termini invenitur

$P^3r+p^3R-3PQr-3pqR+pqPQ+3Rr$, & ſic deinceps.

Sit v radix quæſitæ æquationis & quæſita æquatio erit

$$v^{nm} - Ppv^{nm-1} \left.\begin{matrix} +P^2q-2Qq \\ +p^2Q \end{matrix}\right| \times v^{nm-2} \left.\begin{matrix} -P^3r+3PQr \\ -p^3R+3pqR-3Rr \\ -pqPQ \end{matrix}\right| \times v^{nm-3}, \text{\&c.} = 0.$$

CAP.

CAP. II.

LEMMA III.

SINT omnes radices datæ æquationis poſſibiles invenire numeros radicibus quam proxime æquales.

Reg. 1. Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5}, \&c. = 0.$$

& ſit α major quam β, β quam γ, γ quam δ, & ſic deinceps, &

$\overline{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \&c.}^{\frac{1}{2m}}$ non major eſt quam $n^{\frac{1}{2m}}\alpha$, &

$$\left(\frac{\alpha^{2m}\times\beta^{2m} + \alpha^{2m}\times\gamma^{2m} + \alpha^{2m}\times\delta^{2m} + \&c. + \beta^{2m}\times\gamma^{2m} + \beta^{2m}\times\delta^{2m} + \&c.}{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \&c.}\right)^{\frac{1}{2m}}$$

non major eſt quam $\overline{n-1}|^{\frac{1}{2m}} \times \beta$, &

$$\left(\frac{\alpha^{2m}\times\beta^{2m}\times\gamma^{2m} + \alpha^{2m}\times\beta^{2m}\times\delta^{2m} + \alpha^{2m}\times\gamma^{2m}\delta^{2m} + \&c. + \beta^{2m}\times\gamma^{2m}\times\delta^{2m}}{\alpha^{2m}\times\beta^{2m} + \alpha^{2m}\times\gamma^{2m} + \alpha^{2m}\times\delta^{2m} + \&c. + \beta^{2m}\times\gamma^{2m} + \beta^{2m}\times\delta^{2m} + \&c.}\right)^{\frac{1}{2m}}$$

non major eſt quam $\overline{n-2}|^{\frac{1}{2m}} \times \gamma$, & ſic deinceps obſervatâ progreſſionis lege.

2. Quantitas $\left(\frac{\alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m} + \&c.}{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \&c.}\right)^{\frac{1}{2m}}$ eſt approximatio ad maximam radicem (α) &

$$\overline{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \&c. - \overline{\alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m}}^{\frac{1}{2}}}^{\frac{1}{2m}}$$

eſt approximatio ad radicem (β) & ſic deinceps.

3. Datâ æquatione

$$x^n - * + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + ux^{n-6} - vx^{n-7} + wx^{n+8} - \&c. = 0.$$

cujus ſecundus terminus deeſt & ſint ejus radices

$-\alpha, +\beta, +\gamma, +\delta, +$ &c. & $-\overline{\alpha^8 + \beta^8 + \gamma^8 + \delta^8 + \&c.}^{\frac{1}{8}}$ non major eſt quam $-2^{\frac{1}{8}}\times\alpha$, & conſequenter haud differt a radice ($-\alpha$) per decimam ejus partem, ſed per lemma primum

$$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 + \&c. = 2q^4 - 8qr^2 + 4s^2 - 8q^2s - 8w + 8qu + 8rt;$$

ergo, $\overline{2q^4 - 8qr^2 + 4s^2 - 8q^2s - 8w + 8qu + 8rt}^{\frac{1}{8}}$ negative ſumpta haud differt a radice ($-\alpha$) per decimam ejus partem, & ſic de reliquis.

Ex

Ex methodo ſerierum, erui & demonſtrari poſſunt infinitæ conſimiles propoſitiones.

Et eodem fere modo inveniri poſſunt approximationes ad radices datæ æquationis, quæ habet impoſſibiles radices.

4. Radices æquationis

$$nx^{n-1} - \overline{n-1} \times px^{n-2} + \overline{n-2} \times qx^{n-3} - \&c. = 0.$$

erunt limites inter α & β, β & γ, γ & δ, &c. reſpective, & radices æquationis

$$n \times \overline{n-1}\, x^{n-2} - \overline{n-1} \times \overline{n-2} \times px^{n-2} + \overline{n-2} \times \overline{n-3}\, qx^{n-4} - \&c. = 0.$$

erunt limites inter α & γ, β & δ, γ & ε, &c. & radices æquationis

$$n \times \overline{n-1} \times \overline{n-2}\, x^{n-3} - \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times px^{n-4} + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times qx^{n-5} - \&c. = 0.$$

erunt limites inter α & δ, β & ε, &c. reſpective.

Sequuntur quædam regulæ, e quibus quot radices impoſſibiles ſunt, cognoſci poteſt.

REG. 1. Data æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, invenire, utrum ullas habet impoſſibiles radices, necne.

Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis; per problema primum transformetur data æquatio in alteram, cujus radices (v) erunt reſpective $\alpha^2 + \beta^2 - 2\alpha\beta$, $\alpha^2 + \gamma^2 - 2\alpha\gamma$, $\alpha^2 + \delta^2 - 2\alpha\delta$, &c. $\beta^2 + \gamma^2 - 2\beta\gamma$, $\beta^2 + \delta^2 - 2\beta\delta$, &c. $\gamma^2 + \delta^2 - 2\gamma\delta$, &c.

Æquatio reſultans eſt

$$v^{n \times \frac{n-1}{2}} \overline{- \overline{n-1} \times p^2 - 2nq}\, v^{n \times \frac{n-1}{2} - 1} + \overline{\overline{n-1} \times \frac{n-2}{2} p^4 - 2n \times \overline{n-2}\, p^2 q - 2 \times \overline{n-3}\, pr + 2ns + \overline{2n+3} \times \overline{n-2}\, q^2} \times v^{n \times \frac{n-1}{2} - 2} - \&c. = 0.$$

Si mutentur hujuſce æquationis reſultantis ſigna alternatim de + in − & − in +; nullam radicem impoſſibilem habet data æquatio, ſin hujuſce æquationis ſigna haud continuo mutantur de + in − & − in +, impoſſibiles radices habet data æquatio.

EXEMPLUM.

Sit data æquatio $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, & æquatio reſultans erit, $v^6 - 20v^5 + 142v^4 - 460v^3 + 713v^2 - 520v + 144 = 0$.

 Hujuſce

Hujuſce æquationis ſigna mutantur alternatim de + in — & — in +, & conſequenter nullam habet impoſſibilem radicem data æquatio.

Sit æquatio biquadratica ($x^4 + qx^2 - rx + s = o$) & ſolutionem paullulum diverſam admittit.

Sint $\alpha, \beta, \gamma, \delta$ ejus radices, inveniatur æquatio cujus radices (v) ſunt $\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\gamma-2\beta\delta$, $\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\delta-2\beta\gamma$, $\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\beta-2\gamma\delta$,

Æquatio reſultans eſt

$$v^3 + 8qv^2 + 16 \times \overline{q^2 - s}\, v + 32q^3 - 24qs - 8r^2 = o.$$

Sint q & $32q^3 - 24qs - 8r^2$ reſpective negativæ quantitates, $q^2 - s$ vero affirmativa & omnes datæ biquadraticæ æquationis radices erunt poſſibiles, ſin aliter, tum vel duæ vel quatuor ſunt impoſſibiles.

Reg. 2. Invenire, utrum data æquatio

$$x^{2n} - px^{2n-1} + qx^{2n-2} - rx^{2n-3} + \&c. = o.$$

poſſibilem habet radicem, necne.

Fingetur $v^{2n-1} - \frac{2n-1}{2n} pv^{2n-2} + \frac{2n-2}{2n} qv^{2n-3} - \frac{2n-3}{2n} rv^{2n-4} \&c. = o.$

&

$$v^{2n} - pv^{2n-1} + qv^{2n-2} - rv^{2n-3} + \&c. = w.$$

transformentur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (v) & reſultat æquatio, cujus radix eſt (w) quæ ſit

$$w^{2n-1} - Pw^{2n-2} + Qw^{2n-3} - Rw^{2n-4} + \&c. = o;$$

mutentur hujuſce æquationis terminorum ſigna viciſſim de + in — & — in +; i. e. ſint omnes ejus coefficientes (P, Q, R, S, &c.) affirmativæ & nullum habet radicem poſſibilem data æquatio, & ſic deinceps.

EXEMPLUM.

Sit data æquatio biquadratica, $x^4 - 14x^2 + 24x + 37 = o$.

Fingetur $v^3 - 7v + 6 = o$ & $v^4 - 14v^2 + 24v + 37 = w$ unde æquatio reſultans erit

$$w^3 - 13w^2 - 5280w + 172800 = o.$$

Terminorum hujuſce æquationis ſigna ſeſe ſequuntur hoc ordine + — — +.

Secundi termini ſignum variatur a primi termini ſigno, ſed tertii termini ſignum haud variatur a ſecundi ſigno & conſequenter data æquatio habet poſſibilem radicem.

co-

COROLLARIUM.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$.

Fingetur eodem modo, quo prius docetur, &

$$v^{n-1} - \frac{n-1}{n} pv^{n-2} + \frac{n-2}{n} qv^{n-3} - \&c. = 0.$$

$$v^n - pv^{n-1} + qv^{n-2} - rv^{n-3} + \&c. = w.$$

transformentur hæ duæ æquationes in unam ita ut exterminetur incognita quantitas (v), & æquatio reſultans ſit

$$w^{n-1} - Pw^{n-2} + Qw^{n-3} - Rw^{n-4} + \&c. = 0.$$

Sub primo hujuſce æquationis termino colloca ſignum +, ſub quolibet reliquorum terminorum ſcribe ſignum — vel +, prout ejus ſignum eſt idem vel diverſum a ſigno termini præcedentis, his denique adjiciendum eſt ſignum + vel —, prout (n) eſt par vel impar numerus & fere tot erunt radices poſſibiles, quot ſunt in ſubſcriptorum ſignorum ſerie mutationes de + in —, & — in +.

EXEMPLUM.

Sit æquatio $x^3 - 9x + 20 = 0$.

Fingetur $v^2 - 3 = 0$, & $v^3 - 9v + 20 = w$,
unde æquatio reſultans erit

$$w^2 - 40w + 373 = 0$$
$$+ \quad + \quad + \quad -$$

Sub primo hujuſce æquationis termino (w^2) colloco ſignum +, dein quoniam ſignum (—) ſecundi termini ($-40w$) diverſum eſt a ſigno (+) primi termini ſub termino ($-40w$) colloco ſignum +; & eâ ratione ſub tertio termino colloco ſignum +, his denique adjeci ſignum —, quoniam n eſt impar numerus ſubſcriptorum ſignorum, quæ in hâc ſerie + + + — ſint, una ſolummodo eſt mutatio de + in —; & conſequenter una ſolummodo erit poſſibilis radix.

Hæc regula poſtulat, ut hæ duæ æquationes

$$v^{n-1} - \frac{n-1}{n} pv^{n-2} + \frac{n-2}{n} qv^{n-3} - \frac{n-3}{n} rv^{n-4} + \frac{n-4}{n} sv^{n-5} - \&c. = 0$$

$$v^n - pv^{n-1} + qv^{n-2} - rv^{n-4} + sv^{n-5} - tv^{n-6} + \&c. = w$$

in unam transformentur, ita ut exterminetur incognita quantitas (v.)

Si hoc ſit, tum æquatio reſultans erit

$$\overline{w^{n-1} + \frac{\overline{n-1}^{\,n-1}}{n^n}\,p^n - \frac{\overline{n-1}^{\,n-2}}{n^{n-2}}\times qp^{n-2} + \frac{\overline{n-1}^{\,n-3}}{n^{n-3}}\,rp^{n-3} - }$$

$$\left.\begin{array}{l} \frac{\overline{n-1}^{\,n-4}}{n^{n-4}}\,sp^{n-4} \qquad + \frac{\overline{n-1}^{\,n-5}}{n^{n-5}}\,tp^{n-5} - \\ + \\ \frac{1}{2}n\times\frac{\overline{n-2}^{\,2}}{n^2}\,\frac{\overline{n-1}^{\,n-4}}{n^{n-4}}\,q^2 - n\times\frac{n-2\times n-3}{n^2}\times\frac{\overline{n-1}^{\,n-5}}{n^{n-5}}\,qr + \end{array}\&c.\right] \times w^{n-2} + \&c. = 0.$$

Lex, quam obſervat hæc ſeries exprimens ſecundi termini reſultantis æquationis coefficientem, vel producta literalia, vel coefficientes reſpicit.

Producta literalia eandem, quam producta literalia ſeriei in lemmate ſecundo traditæ, obſervant legem.

Coefficiens literalis producti $p^{n-m}q^{\alpha}r^{\beta}s^{\gamma}t^{\delta}$, &c. erit æqualis fractioni, cujus numerator eſt

$$n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\ldots\overline{n-m}\times\overline{n-2}^{\,\alpha}\times\overline{n-3}^{\,\beta}\times\overline{n-4}^{\,\gamma}\times\overline{n-5}^{\,\delta}\times\&c.\times\frac{\overline{n-1}^{\,n-m}}{n^{n-m}}$$

Denominator vero eſt

$$1\times2\times3\times\ldots\alpha\times1\times2\times3\times\ldots\beta\times1\times2\times3\times4\times\ldots\gamma\times1\times2\times3\times4\times\ldots\delta, \&c.$$

Hic excipiatur primi ſeriei termini coefficiens, quæ ſit $\frac{\overline{n-1}^{\,n-1}}{n^n}$.

Sit $\mathfrak{Q}$ ultimus datæ æquationis terminus & ejus coefficiens eſt $n-1$.

Coefficientes reliquorum reſultantis æquationis terminorum, legem, quæ exprimi poteſt per ſeriem haud obſervant, ſed hic obſervandum eſt, ut in omnibus litera (p) nunquam habet plures quam (n) dimenſiones.

REG. 3. Sint $\alpha, \beta, \gamma, \delta$, &c. ſucceſſivæ datæ æquationis ($x^n - px^{n-1} + qx^{n-2} - \&c. = 0$) radices quarum nullæ ſint inter ſe æquales, ſit (π) major quam maxima radix (α,) ρ limes inter α & β, σ inter β & γ & ſic deinceps.

Subſtituantur limites (π, ρ, σ, τ, &c.) pro incognitâ quantitate (x) in datâ æquatione & tot erunt radices poſſibiles, quot ſunt mutationes

ones ſignorum quantitatum reſultantium de + in — & — in +, cæteræ vero impoſſibiles ſunt.

Sed in omni caſu inveniri poſſunt limites (ϱ, σ, τ, &c.) per ſubſequentem methodum: transformetur data æquatio in alteram, cujus radices ſint $\frac{1}{\alpha-\beta}$, $\frac{1}{\alpha-\gamma}$, $\frac{1}{\alpha-\delta}$, &c. $\frac{1}{\beta-\alpha}$, $\frac{1}{\beta-\gamma}$, $\frac{1}{\beta-\delta}$, &c. $\frac{1}{\gamma-\alpha}$, $\frac{1}{\gamma-\beta}$, $\frac{1}{\gamma-\delta}$, &c $\frac{1}{\delta-\alpha}$, &c. reſpective, ſit $\left(\frac{1}{A}\right)$ major quam maxima æquationis reſultantis radix & erit (A) minor vel quam differentia inter α & β, vel β & γ, γ & δ, &c.

Inventis igitur quantitate (π) quæ major eſt quam maxima radix, et quantitate (A) quæ minor eſt quam minima differentia inter duas ſucceſſivas radices; continuo auferantur A, $2A$, $3A$, &c. reſpective a π & tandem reſtabunt limites inter ſucceſſivas radices $\alpha, \beta, \gamma, \delta$, &c.

Reg. 4. Data æquatione, invenire quot ſunt ejus radices impoſſibiles & quot poſſibiles.

Duæ radices primo fiunt æquales, deinde impoſſibiles, & conſequenter regula, quæ invenit, quando duæ radices fiunt æquales, invenit etiam, quando vel duæ, vel ſex vel decem fiunt impoſſibiles.

Duæ radices bis, ter, &c. fiunt æquales, deinde quatuor, ſex, &c. impoſſibiles; & conſequenter regulæ, quæ inveniunt, quando duæ radices bis, ter &c. fiunt æquales, inveniunt etiam quando quatuor &c. ſex &c. fiunt impoſſibiles.

Per ſubſequens caput inveniri poteſt, quando duæ radices, ſemel, bis, ter &c. fiunt æquales & conſequenter quando duæ, quatuor, ſex &c. fiunt impoſſibiles.

EXEMPLUM.

Sit æquatio biquadratica $x^4+qx^2-rx+s=0$ invenire quot ſunt ejus radices impoſſibiles.

Sit $s^3-8q^2s^2+\overline{36r^2q+16q^4}\times s-16r^2q^3-27r^4$ nihilo æqualis & duæ radices erunt æquales, ſit vero negativa quantitas & duæ & non plures radices erunt impoſſibiles, ſit affirmativa quantitas & vel q^2-4s vel $9r^2+8qs-2q^2$ vel $r^4-8qr^2s+16q^2s^2-64s^3$ negativa quantitas & quatuor datæ æquationis radices erunt impoſſibiles.

5.

REG. 5. Datis quatuor ab initio æquationis n dimenſionum ($x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= 0$) terminis ($x^n, px^{n-1}, qx^{n-2}, rx^{n-3}$) invenire, utrum ex iis colligi poteſt æquationem impoſſibiles habere radices.

Sit cubica æquatio ($x^3 - Px^2 + Qx - R = 0$) & duæ ejus radices erunt impoſſibiles, ſi

$4P^3R - P^2Q^2 - 18PQR + 27R^2 + 4Q^3$ ſit negativa quantitas.

In hac quantitate pro P, Q, R, ſcribe

$\frac{np}{3}$, $\frac{n}{2} \times \frac{n-1}{3} q$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} r$, ſi quantitas reſultans ſit negativa, exinde colligi poteſt æquationem impoſſibiles habere radices.

Datâ multarum dimenſionum æquatione & per præcedentes regulas invenire, quot ſunt radices impoſſibiles, nonnunquam requiritur calculus maxime laborioſus; ſunt quidem alii modi, quibus hoc problema ſolvi poteſt & contractiores, at ſæpenumero ne unam radicem quidem impoſſibilem inveniant, quamvis plurimæ ſint, & in altis æquationibus perraro omnes impoſſibiles radices detegant.

REG. 1. Sit æquatio $x^n - px^{n-1} + qx^{n-2}$ &c. $= 0$, transformetur hæc æquatio in alteram, cujus radices ſint quadrata hujuſce æquationis radicum & tot erunt radices impoſſibiles, quot æquationis reſultantis ſignorum in continuâ ſerie ſunt progreſſiones de + in + vel — in —.

REG. 2. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= 0$. Sub primo & ultimo hujuſce æquationis terminis colloca ſignum +, ſub quolibet intermedio termino (x^{n-m}) ſi quadratum ejus ductum in fractionem $\frac{\overline{n-m} \times m}{\overline{n-m+1} \times \overline{m+1}}$ majus eſt quam rectangulum terminorum utrinque conſiſtentium colloca ſignum +, ſin minus, ſignum —; & tot erunt radices impoſſibiles, quot ſunt in ſubſcriptorum ſignorum ſerie mutationes de + in — & — in +.

REG. 3. Datâ æquatione $x^n + qx^{n-2} - rx^{n-3} +$ &c. $= 0$, in quâ deeſt ſecundus terminus.

Sub primo & ultimo hujuſce æquationis termino colloca ſignum +, ſub quolibet reliquorum terminorum (x^{n-m}) ſi quadratum ejus ductum

ductum in fractionem $\frac{\overline{n-m}\times\overline{m+1}^{2}\times\overline{m-1}}{\overline{n-m+1}\times\overline{m+2}\times m^{2}}$ majus eſt quam rectangulum terminorum utrinque conſiſtentium, colloca ſignum + ſin minus ſignum —, & fere tot erunt radices poſſibiles, quot ſunt in ſubſcriptorum ſignorum ſerie mutationes de + in — & — in +.

Modus demonſtrandi maxime accommodatus huic generi, in quo quæritur, quot radices impoſſibiles continentur in data æquatione; eſt multiplicare datam æquationem 1mo. Per $x-a=0$; deinde per $x^2-2ax+a^2+b^2=0$, cum enim prior æquatio unicam radicem habeat, eamque poſſibilem; poſterior autem duas impoſſibiles; manifeſtum eſt eundem futurum eſſe numerum radicum impoſſibilium in æquatione, quæ oritur ex priori multiplicatione ac in data æquatione: æquatio autem, quæ oritur ex ſecunda multiplicatione duas habebit radices impoſſibiles, quæ in datâ æquatione non contineantur.

EXEMPLUM.

Sit data æquatio $x^2-2x+2=0$; quæ ſecundum *Newtoni* regulam duas habet radices impoſſibiles, multiplicatur per $x+2=0$ & fit $x^3-2x+4=0$, quæ ſecundum eandem regulam nullas habet radices impoſſibiles, debet autem habere duas, regula, igitur non ſemper vera eſt.

Modus autem demonſtrationis de quo loquor, non huic quæſtioni ſolummodo proprius eſt; ſed commode adhiberi poteſt, ubicunque quæritur, quot quantitates dati cujuſvis generis in datâ æquatione contineantur.

Exemplum ſit notiſſima *Carteſii* regula "tot eſſe radices affirmati-"vas, quot ſignorum in continua ſerie mutationes de + in — & —"in +; ſi modo nullæ æquationis radices ſunt impoſſibiles."

Si enim data quævis æquatio, multiplicetur per $x-a=0$: mutationum ſignorum numerus augebitur per unitatem, ſi autem eadem æquatio multiplicetur per $x+a=0$, eædem erunt ſignorum mutationes in novâ, ac in datâ æquatione.

Sit

Sit enim data æquatio

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - Fx^{n-6} - \&c. = 0$$

quæducta in $x - a$

dat

$$\begin{array}{l} x^{n+1} - Ax^n - Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + Ex^{n-4} - Fx^{n-5} - \&c. = 0 \\ \quad - a \quad + Aa \quad + aB \quad - aC \quad - aD \quad - aE \quad + \&c. = 0 \end{array}$$

Jam vero quod ad ſigna hujus novæ æquationis attinet, manifeſtum eſt,

1. Terminum quemcunque novæ æquationis idem habere ſignum cum termino datæ æquationis, cui ſubſcribitur, ſi modo coefficiens iſtius termini major ſit quam coefficiens termini antecedentis multiplicata per $(a.)$

Ex. gr. Signum quinti termini in data æquatione eſt affirmativum ſcil $(+D,)$ ſignum autem termini quinti in novâ æquatione eſt $(D - aC)$ quod etiam affirmativum eſt, ſi modo D major eſt quam aC.

2. Si vero coefficiens alicujus termini in datâ æquatione minor ſit quam coefficiens termini antecedentis multiplicata in (a); Tum terminus novæ æquationis, qui termino propoſito ſubſcribitur diverſum habet ſignum a termino antecedente in datâ æquatione.

Sit, ex. gr. coefficiens termini quinti in datâ æquatione minor quam (aC), coefficiens termini antecedentis multiplicata per (a), & $(D - aC)$ coefficiens termini quinti in novâ æquatione eſt negativa, coefficiens autem termini antecedentis in data æquatione eſt affirmativa $+C$.

Ex. gr. Sit igitur B major quam Aa, C quam aB: D autem minor quam aC. Jam vero quia D minor eſt quam aC, erit etiam E minor quam aD, & F quam aE, &c. uſque ad finem æquationis, cum enim omnes radices datæ æquationis ſint poſſibiles, notum eſt

	EC	minorem eſſe quam	D^2
Eſt etiam	D	minor quam	aC
Ergo	E	minor eſt quam	aD.

Et duæ æquationes hoc modo ſcribi poſſunt.

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - Fx^{n-6} \&c. = 0.$$

$$x^{n+1} - \overline{A+a}\,x^n - \overline{B-aA}\,x^{n-1} + \overline{C+aB}\,x^{n-2} - a\overline{C-D}\,x^{n-3} - \overline{aD-E}\,x^{n-4} - a\overline{E+F}x^{n-5} \&c. = 0.$$

Patet

Patet igitur a termino primo ad terminum quartum in utrâque æquatione ſigna eſſe eadem.

A termino autem quarto in datâ æquatione, & a termino quinto in novâ, ſigna eſſe ſemper diverſa.

In utroque caſu igitur eædem ſunt mutationes ſignorum; eſt autem in novâ æquatione a termino quarto ad quintum unica mutatio ſignorum, quæ non eſt in data æquatione.

Eodem modo pateret numerum mutationum ſignorum nec augeri nec minui multiplicatione per $(x+a.)$

Unde ſequitur tot eſſe radices affirmativas, quot ſunt mutationes, ſignorum de + in — & — in +, cæteras vero negativas eſſe.

CAP. III.

PROB. II.

INVENIRE illas datæ æquationis

$$x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + s x^{n-4} - \&c. = 0.$$

Radices ($\alpha, \beta, \gamma, \delta$, &c.) *quæ datam inter ſe habent relationem.*

Scribatur $x - \alpha = 0$, $x - \beta = 0$, $x - \gamma = 0$, $x - \delta = 0$, &c. & per æquationem ex horum continuâ multiplicatione $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta}$, &c. $= 0$. generatam dividatur data æquatio.

Singulus reſidui terminus nihilo æqualis fiat; & æquationum reſultantium quicunque communis diviſor erit radix quæſita.

EXEMPLUM I.

Sint (α & β) duæ radices datæ æquationis inter ſe æquales: & ſit α vel $\beta = a$; ſcribatur $x - a = 0$ & $x - a = 0$, & rectangulum $\overline{x-a} \times \overline{x-a} = 0$ fit $x^2 - 2ax + a^2 = 0$; per hoc rectangulum data dividatur æquatio; & operatio eſt hujuſmodi,

$$x^2 - 2ax + a^2)\, x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0\,(x^{n-2} + \overline{2a-p}\, x^{n-2}\ \&c.$$

$$\underline{x^n - 2ax^{n-1} + a^2 x^{n-2}}$$

$$\overline{2a-p}\, x^{n-1} + \overline{q-a^2}\, x^{n-2}$$

$$\underline{\overline{2a-p}\, x^{n-1} - \overline{4a^2-2ap}\, x^{n-2} + \overline{2a^3-a^2p}\, x^{n-3}}$$

$$\overline{3a^2-2ap+q}\, x^{n-2} - \overline{2a^3-a^2p+r}\, x^{n-3}, \ \&c.$$

Patet, prioris reſidui termini coefficientem eſſe

$$n a^{n-1} - \overline{n-1}\, p a^{n-2} + \overline{n-2}\, q a^{n-3} - \overline{n-3}\, r a^{n-4} + \&c.$$

poſteriorem vero reſidui terminum eſſe

$$\overline{n-1}\, a^n - \overline{n-2}\, p a^{n-1} + \overline{n-3}\, q a^{n-2} - \overline{n-4}\, r a^{n-3}\ \&c.$$

Fiant hæ duæ quantitates nihilo reſpective æquales, & duarum æquationum reſultantium quicunque communis diviſor erit radix quæſita.

EX-

EXEMPLUM II.

Sint (m) radices $\alpha, \beta, \gamma, \delta$, &c. datæ æquationis inter ſe æquales.

Pro unâ radicum ($\alpha, \beta, \gamma, \delta$, &c. ſcribatur ($a$) & erit $x-a=0$, $x-a=0$, $x-a=0$, &c. ducantur hæ quantitates continuo in ſeſe, & fit contentum

$$\overline{x-a}^{\,m} = x^m - max^{m-1} + m \cdot \frac{m-1}{2} a^2 x^{m-2} - m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} a^3 x^{m-3} \,\&c. = 0$$

per hoc contentum data dividatur æquatio, & operatio eſt,

$$x^m - max^{m-1} + m \cdot \tfrac{m-1}{2} a^2 x^{m-2} - \&c.)\; x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0 \; (x^{n-m} + \overline{ma-p}\, x^{n-m-1} + \&c.$$

$$x^n - max^{n-1} + m \cdot \tfrac{m-1}{2} a^2 x^{n-2} - \&c.$$

$$\overline{ma-p}\, x^{n-1} + \overline{q - m \times \tfrac{m-1}{2} a^2}\, x^{n-2} + \&c.$$

$$\overline{ma-p}\, x^{n-1} + \overline{map - m^2 a^2}\, x^{n-2} + \&c.$$

$$\overline{q - map + m \cdot \tfrac{m+1}{2} a^2}\, x^{n-2} - \&c.$$

Patet coefficientem primi reſidui termini eſſe

$$m \times \tfrac{m+1}{2} \times \tfrac{m+2}{3} \ldots \tfrac{n}{n-m+1} a^{n-m+1} - m \times \tfrac{m+1}{2} \ldots \tfrac{n-1}{n-m} pa^{n-m+2} + m \cdot \tfrac{m+1}{2} \ldots \times \tfrac{n-2}{n-m-1} qa^{n-m+3} - m \ldots \tfrac{n-3}{n-m-2} ra^{n-m+4} \,\&c.$$

Secundi autem reſidui termini coefficientem eſſe

$$\overline{n-m+1} \times \overline{m-1} \times \tfrac{m}{2} \times \tfrac{m+1}{3} \ldots \tfrac{n}{n-m+2} a^{n-m+2} - \overline{n-m} \times \overline{m-1} \times \tfrac{m}{2} \times \tfrac{m+1}{3} \ldots \tfrac{n-1}{n-m+1} pa^{n-m+1} + \overline{n-m-1} \times \overline{m-1} \times \tfrac{m}{2} \times \tfrac{m+1}{3} \ldots \tfrac{n-2}{n-m} qa^{n-m} + \&c.$$

Tertii reſidui termini coefficientem eſſe

$$\overline{n-m+1} \times \tfrac{n-m-2}{2} \times \overline{m-2} \times \tfrac{m-1}{2} \times \tfrac{m}{3} \times \tfrac{m+1}{4} \ldots \tfrac{n}{n-m+3} a^{n-m+3} - \overline{n-m} \times \tfrac{n-m+1}{2} \times \overline{m-2} \times \tfrac{m-1}{2} \times \tfrac{m}{3} \ldots \tfrac{n-1}{n-m+2} pa^{n-m+2} + \overline{n-m-1} \times \tfrac{n-m}{2} \cdot \tfrac{\overline{m-2}}{1} \ldots \tfrac{n-2}{n-m+1} qa^{n-m+2}$$

Quarti reſidui termini coefficientem eſſe

$$\overline{n-m+1} \times \tfrac{n-m+2}{2} \times \tfrac{n-m+3}{3} \times \overline{m-3} \times \tfrac{m-2}{2} \times \tfrac{m-1}{3} \ldots \tfrac{n}{n-m+4} a^{n-m+4} - \overline{n-m} \times \tfrac{n-m+1}{2} \times \tfrac{n-m+2}{3} \times \overline{m-3} \times \tfrac{m-2}{2} \ldots \tfrac{n-1}{n-m+3} pa^{n-m+3} + \overline{n-m+1} \times \tfrac{\overline{n-m}}{2} \times \tfrac{n-m+1}{3} \times m-2 \ldots \tfrac{n-2}{n-m+2} qa^{n-m+1} \,\&c.$$

& ſic deinceps.

Fiant hæ coefficientes nihilo reſpective æquales & æquationum reſultantium communis diviſor erit radix quæſita.

EXEMPLUM III.

Sint radices ($\alpha, \beta, \gamma, \delta$, &c.) reſpective arithmeticæ progreſſionis (m) termini ($a, a+b, a+2b, a+3b, \ldots a+\overline{m-1}\,b$.)

Dicantur ſumma numerorum ($1, 2, 3, 4, 5 \ldots m-1$,) & aggregata contentorum ſub ſingulis binis, ternis, quaternis, &c. reſpective P, Q, R, S, T, & multiplicentur radices

$$\overline{x-a} \times \overline{x-a-b} \times \overline{x-a-2b} \ldots \overline{x-a-\overline{m-1}\,b}$$

continuo in ſeſe, & contentum erit

$$\begin{array}{llll}
x^m - max^{m-1} & + m \times \frac{m-1}{2} a^2 x^{m-2} & - m \times \frac{m-1}{2} \times \frac{m-2}{3} a^3 x^{m-3} \\
\quad\;\; - Pb & + \overline{m-1}\, Pba & - m-1 \times \frac{m-2}{2} Pba^2 \\
 & + \quad Qb^2 & - \quad \overline{m-2}\, Qb^2 a \\
 & & - \quad Rb^3
\end{array}$$

$$\begin{array}{ll}
+ & m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^4 x^{m-4} \text{ \&c.} = 0, \\
+ & \overline{m-1} \times \frac{m-2}{2} \times \frac{m-3}{3} Pba^3 \\
+ & \overline{m-2} \times \frac{m-3}{2} Qb^2 a^2 \\
+ & \overline{m-3} \times Rb^3 a \\
 & Sb^4
\end{array}$$

Per hoc contentum data dividatur æquatio, & reſidui terminorum quicunque communis diviſor erit radix quæſita.

EXEMPLUM IV.

Sint radices ($\alpha, \beta, \gamma, \delta$, &c.) geometricæ progreſſionis ($a, ra, r^2 a \ldots r^{m-1} a$) ($m$) termini.

Multiplicentur factores $\overline{x-a} \times \overline{x-ar} \times \overline{x-ar^2} \ldots \overline{x-ar^{m-1}}$ continuo in ſeſe, & contentum erit

$$x^m - \frac{a-ar^m}{1-r} x^{m-1} + \frac{a^2 \times \overline{r-r^m-r^{m+1}+r^{2m}}}{\overline{1-r} \times \overline{1-r^2}} x^{m-2} - \text{\&c.} = 0.$$

Coefficiens termini (x^{m-s}) hujus æquationis erit æqualis fractioni, cujus numerator eſt

$$a^s r^{s \times \frac{s-1}{2}} \times \overline{1-r^{m-s+1}} \times \overline{1-r^{m-s+2}} \times \overline{1-r^{m-s+3}} \ldots \times \overline{1-r^m}$$

De-

Denominator vero $\overline{1-r} \times \overline{1-r^2} \times \overline{1-r^3} \dots \overline{1-r^t}$.

Per hoc contentum data dividatur æquatio, & terminorum reſidui communis diviſor erit radix quæſita.

Eſt & alia methodus & ſæpe multo contractior inveniendi radices datæ æquationis datam inter ſe habentes relationem.

1. Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationes, datam inter ſe habentes relationem, ſcribantur pro incognitâ datæ æquationis quantitate (x) reſpective $\alpha, \beta, \gamma, \delta$, &c. & æquationum reſultantium communis diviſor erit una radicum, quæ datam habent relationem ad ſe invicem.

EXEMPLUM I.

Sint a, β, γ, δ, &c. reſpective, $a, a+b, a+2b$, arithmeticæ progreſſionis termini & ſcribantur pro incognitâ quantitate (x) reſpective $a, a+b, a+2b$, &c. & reſultant æquationes.

$a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} +$ &c. $= 0$.

$\overline{a+b}^{n} - p \times \overline{a+b}^{n-1} + q\overline{a+b}^{n-2} - r\overline{a+b}^{n-3} +$ &c. $= 0$.

$\overline{a+2b}^{n} - p\overline{a+2b}^{n-1} + q\overline{a+2b}^{n-2} - r\overline{a+2b}^{n-3} +$ &c. $= 0$.

Harum æquationum communis diviſor, erit radix quæſita.

Hinc patet tertium & quartum præcedentis methodi exemplum ex hâc quam priori methodo multo contractiorem habere ſolutionem.

COROLLARIUM.

Omnis impoſſibilis radix datæ æquationis hujuſce formulæ $a+\sqrt{-b^2}$, alteram habet ſibi ipſi correſpondentem $a-\sqrt{-b^2}$, quibus compoſitis reſultat quadraticus diviſor $x^2 - 2ax + a^2 + b = 0$.

Subſtituatur enim in data æquatione pro incognita quantitate (x) & $a+\sqrt{-b^2}$ & $a-\sqrt{-b^2}$; & conſtabit ſi modo $a+\sqrt{-b^2}$ ſit datæ æquationis radix, $a-\sqrt{-b^2}$ etiam eſſe datæ æquationis radicem.

2. Si data relatio inter m radices $(\alpha, \beta, \gamma, \delta$, &c.) datæ æquationis,

$(x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= 0)$

ſolummodo exprimi poteſt per æquationem hujuſmodi,

$x^m - Px^{m-1} + Qx^{m-2} - Rx^{m-3}$, &c. $= 0$,

tum

tum aſſumatur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= w$, & problematis primi ope, inveniatur æquatio, cujus radix eſt w, quæ ſit $w^m - aw^{m-1} + bw^{m-2} - cw^{m-3} +$ &c. $= 0$.

Fiant coefficientes a, b, c, &c. nihilo reſpective æquales & æquationum reſultantium communis diviſor eſt radix quæſita.

Eſt etiam alia methodus inveniendi radices datæ æquationis datam inter ſe habentes relationem.

Sint (m) radices $\alpha, \beta, \gamma, \delta$, &c. datam inter ſe habentes relationem, ducantur continuo in ſeſe factores $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta}$, &c. & contentum multiplicetur in quantitatem hujus generis

$$x^{n-m} - Ax^{n-m-1} + Bx^{n-m-2} - \text{\&c.} = 0$$

& reſultantis producti coefficientes fiant æquales datæ æquationis coefficientibus correſpondentibus, æquationum reſultantium communis diviſor erit radix quæſita.

EXEMPLUM

Sint (m) radices $\alpha, \beta, \gamma, \delta$, &c. inter ſe æquales & ſit

$$a = \alpha = \beta = \gamma = \delta, \text{ \&c.}$$

tum contentum ex continuâ multiplicatione factorum

$$\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma}, \text{ \&c.}$$

erit $\overline{x-a}^m = x^m - max^{m-1} + m.\frac{m-1}{2}a^2x^{m-2} -$ &c. $= 0$, & eorum contentum multiplicetur in aſſumptam quantitatem

$$x^{n-m} - Ax^{n-m-1} + Bx^{n-m-1} - \text{\&c.} = 0, \text{ \& productum erit}$$

$$x^n - \overline{A + ma}x^{n-1} + \overline{B + m.\frac{m-1}{2}a^2}x^{n-2} - \overline{C + m.\frac{m-1}{2}.\frac{m-2}{3}a^3}x^{n-3} \text{\&c.} = 0.$$

Fiant coefficientes hujuſce æquationis æquales datæ æquationis correſpondentibus coefficientibus.

$$A + ma = p,\ B + m.\frac{m-1}{2}a^2 = q,\ C + m.\frac{\overline{m-1}}{2}.\frac{\overline{m-2}}{3}a^3 = r$$

& harum æquationum communis diviſor erit radix quæſita.

PROB. III.

Datâ æquatione

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} - \text{\&c.} = 0,$$

invenire ejus rationales diviſores.

1. Sint

1. Sint p, q, r, s, &c. integri numeri, & pro incognitâ quantitate (x) ſubſtitue tres vel plures hujus arithmeticæ progreſſionis terminos $-1, 0, 1, 2$, &c. & numerorum reſultantium ſint $z+1, z, z-1$, &c. reſpective diviſores, quorum ſit (z) diviſor quantitatis reſultantis e ſubſtitutione termini (0) & $x-z$ erit quæſitus diviſor.

2. Eodem modo transformetur data æquatio in aliam, cujus radices ſunt quæcunque rationalis functio datæ æquationis radicum, ultimi & datæ & reſultantis æquationis termini ſint α & π diviſores reſpective, & (π) erit ea rationalis functio diviſoris (α), ſimodo α ſit rationalis radix quæſita.

Eſt alia methodus rationales datæ æquationis

$$S+Rx+\mathcal{Q}x^2+Px^3+Ox^4\ldots x^n=0$$

in quâ $S, R, \mathcal{Q}, P$, &c. integri ſunt numeri, radices inveniendi.

1. Sit coefficiens termini (x^n) in quo dimenſiones incognitæ quantitatis (x) inveniuntur maximæ; unitas, & rationalis radix quæſita (a): tum $\frac{S}{a}$, $\frac{Ra+S}{a^2}$, $\frac{\mathcal{Q}a^2+Ra+S}{a^3}$, $\frac{Pa^3+\mathcal{Q}a^2+Ra+S}{a^4}$, $\frac{Oa^4+Pa^3+\mathcal{Q}a^2+Ra+S}{a^5}$, &c. integri ſunt numeri.

EXEMPLUM.

Sit data æquatio $-35-44x+7x^2-7x^3+x^4=0$.

Pro $S, R, \mathcal{Q}, P, O$, ſcribantur reſpective $-35, -44, 7, -7, 1$.

Coefficientis ($S, -35$) diviſores ſunt $1, 5, 7, -1, -5, -7$, pro rationali radice quæſita (a) aſſumatur diviſor (7), & $\frac{S}{a}=-\frac{35}{7}=-5$, $\frac{Ra+S}{a^2}=-\frac{5-44}{7}=-7$, $\frac{\mathcal{Q}a^2+Ra+S}{a^3}=\frac{7-7}{7}=0$. $\frac{Pa^3+\mathcal{Q}a^2+Ra+S}{a^4}=\frac{-7+0}{7}=-1$; unde ſequitur numerum (7) eſſe rationalem datæ æquationis radicem.

2. Sit coefficiens termini (x^n) quicunque integer numerus, cujus ſit (m) diviſor & $\left(\frac{a}{m}\right)$ quæſita radix; tum $\frac{S}{a}$, $\frac{Ra+mS}{a^2}$ $\frac{\mathcal{Q}a^2+mRa+m^2S}{a^3}$, $\frac{Pa^3+m\mathcal{Q}a^2+m^2Ra+m^3S}{a^4}$ & ſic deinceps, integri erunt numeri.

Si

Si nullus hac methodo occurrit diviſor, concludendum eſt æquationem non admittere ſimplicem diviſorem, poteſt tamen fortaſſe habere quadraticum diviſorem, qui ſit $mx^2 - bx + a$, in quo (m) eſt diviſor coefficientis datæ æquationis termini (x^n); & (a) diviſor ultimi datæ æquationis termini (S); & $m - b + a$ diviſor quantitatis

$$S + R + Q + P + C, \text{\&c.}$$

tum $\frac{Ra + bS}{a^2}$, $\frac{Qa^2 - mSa + bRa + b^2 S}{a^3}$

$\frac{Pa^3 + bQa^2 - mRa^2 - 2mbSa + b^2 Ra + b^3 Sa}{a^4}$ &c. integri erunt numeri, & ſic deinceps.

PROB. IV.

Invenire datæ æquationis duas incognitas quantitates (x & y) *habentis, rationales radices ſibi ipſis correſpondentes.*

1. Sit data æquatio $\alpha x - \beta y = P$ in quâ incognitæ quantitates $(x$ & $y)$ unam ſolummodo habent dimenſionem, invenire earum correſpondentes radices, quæ ſunt integri numeri.

Sit (α) minor coefficiens, & dividantur duo reliqui termini $(-\beta y$ & $P)$ per minorem coefficientem (α), & ſit reſiduum $\Gamma y = Q$: ſupponatur $\alpha v - \Gamma y = Q$ & in hâc novâ æquatione reſultante ſit operatio eadem ac in datâ; & ſic deinceps, uſque dum reſiduum nihilo fiat æquale.

Deinde pro ultimæ æquationis quantitatibus aſſumantur quicunque integri numeri & quod prius ſubductum fuit, addatur, &c. & invenientur incognitarum quantitatum $(x$ & $y)$ correſpondentes integri valores.

EXEMPLUM.

Sit $7x = 16y + 20$ dividatur $(16y + 20)$ per (7) & reſiduum eſt $2y + 6 = 7v$: dividatur hæc æquatio per (2) minorem coefficientem & reſiduum eſt $v = 2w$. Pro (w) aſſumatur quicunque integer numerus & $2w = v$ & $2y = 7v - 6 = 14w - 6$

unde $y = 7w - 3$ &

$$x = \frac{16y + 20}{7} = \frac{112w - 28}{7} = 16w - 4.$$

Et ſic de æquatione plures incognitas quantitates habente.

2. Sit data æquatio duas incognitas quantitates (x & y) continens, quæ habent duas vel plures dimenſiones.

Invenire rationales incognitarum quantitatum (x & y) radices.

1mo. Sit data æquatio (n) dimenſionum, pro incognitis quantitatibus x & y in data æquatione ſubſtituantur reſpective

$$(x=)Az^{n}+Bz^{n-1}+Cz^{n-2}+ \&c. \quad \&$$
$$(y=)az^{n}+bz^{n-1}+cz^{n-2}+ \&c.$$

vel

$$(x=)\frac{Az^{n}+Bz^{n-1}+Cz^{a-2}+ \&c.}{Pz^{n}+Qz^{n-1}+Rz^{n-2}+ \&c.}$$
$$(y=)\frac{az^{n}+bz^{n-1}+cz^{n-2}+ \&c.}{Pz^{n}+Qz^{n-2}+Rz^{n-2}+ \&c.}$$

Æquationis reſultantis ſupponantur ſinguli termini nihilo æquales, & exinde inveniri poſſunt rationales coefficientium (A, B, C, &c. a, b, c, &c.) valores.

EXEMPLUM.

Sit æquatio $4y^{2}-4xy+x^{2}+5x-11y+8=0$. Scribantur pro incognitis quantitatibus (x & y) reſpective

$$(y=)\ z^{2}+Bz+C \ \&$$
$$(x=)az^{2}+bz+c \ \& \text{ reſultat æquatio}$$

$$\overline{4-4a+a^{2}}\times z^{4}+\overline{8B-4aB-4b+2ab}\,x^{3}+\overline{8C-4c-4aC}$$
$$\overline{+2ac+5a+4B^{2}-4Bb+b^{2}-11}\,z^{2}+\overline{8BC-4bC-4Bc+2cb}$$
$$\overline{+5b-11B}\,z+4C^{2}-4Cc+c^{2}+5c-11C+8=0.$$

Fiant hujus æquationis reſultantis coefficientes nihilo reſpective æquales: & invenientur coefficientium (a, b, c, B, C) valores reſpective ($2, 1, 1, 1, 2$): ſubſtituantur hi numeri pro ſuis valoribus, & fit

$$y=z^{2}+z+2 \ \&$$
$$x=2z^{2}+z+1.$$

COROLLARIUM.

Sint x & y incognitæ datæ æquationis quantitates &

$$x=Az^{n}+Bz^{n-1}+Cz^{n-2}+Dz^{n-3}+$$
$$y=az^{n}+bz^{n-1}+cz^{n-2}+dz^{n-3}+ \&c.$$

Aſſumatur æquatio

$$Ay^{m}+\overline{a+bx}\,y^{m-1}+\overline{c+dx+ex^{2}}\,y^{m-2}+ \&c.=0.$$

ita

ita ut ſit productum $\left(m \times \frac{\overline{m+3}}{2}\right)$ æquale vel majus quam $(nm+1)$: ſubſtituantur pro incognitis quantitatibus (x & y) earum reſpectivi valores, æquationis reſultantis omnium terminorum ſingulæ coefficientes nihilo æquales fiant, & ſi $\left(m \times \frac{\overline{m+3}}{2}\right)$ majus eſt quam $(nm+1,)$ tum infinitæ, ſin æquales, tum una ſolummodo deduci poteſt æquatio, in quâ incognitæ quantitates (x & y) ſunt reſpective

$$Az^n + Bz^{n-1} + Cz^{n-2} + \&c.$$
$$\& \; az^n + bz^{n-1} + cz^{n-2} + \&c.$$

Eodem modo, ſæpe dato uno, duobus vel pluribus correſpondentibus valoribus incognitarum quantitatum (x & y) inveniri poſſunt alii innumeri.

Si nullæ rationales radices hâc methodo occurrunt, concludendum eſt datam æquationem haud admittere generalem regulam, quæ inveniat omnes incognitarum quantitatum rationales radices: ſed fortaſſe habent incognitæ quantitates innumeras rationales radices ſibi ipſis correſpondentes, & juxta *Walliſii* approximationis methodum ſemper inveniri poſſunt.

Sæpe vero in hujuſmodi problematibus ſolvendis tentando vel conjecturâ ſolummodo procedere poſſimus; vix enim ullam aliam memethodum præbet natura.

Sunt etiam hujuſmodi theoremata, quorum demonſtrationes invenire nunquam valebit humani ingenii vis; ſi vero demonſtrationem admittunt quævis propoſitiones, ea ex præcedentibus principiis petenda eſt.

CAP. IV.

PROB. V.

INVENIRE æquationis dimenſiones, cujus ope altera ($x^n - px^{n-1} + qx^{n-2}$, &c. $= o$) n *dimenſionum in alteram* ($x^m - ax^{m-1} + bx^{m-2}$, &c. $= o$) m *dimenſionum reduci poteſt, poſitâ* n *majore quam* m, *& radicibus reductæ æquationis* $x^m - ax^{m-1} + bx^{m-2}$ &c. $= o$, *etiam eſſe radices æquationis* $x^n - px^{n-1} + qx^{n-2}$, &c. $= o$, *ex quâ reducta fuit.*

Dimenſiones requiſitæ erunt æquales fractioni

$$n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-m+1}{m} .$$

DEMONSTRATIO.

Æquatio $x^n - px^{n-1} + qx^{n-2}$, &c. $= o$, habet (n) radices, & ex quâvis combinatione (m) radicum hujus æquationis formari poteſt æquatio m dimenſionum, quæ reſpondebit problemati; ergo quot ſunt combinationes m radicum, in majore multitudine (n) radicum, tot erunt problematis ſolutiones & conſequenter tot erunt radices æquationis reducentis.

Numerus autem combinationum, quibus quantitatum vel radicum multitudo (m) in majori multitudine (n) eſſe poſſit; æqualis erit fractioni $n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-m+1}{m}$, † ergo numerus radicum & conſequenter dimenſiones æquationis reducentis æqualis erit fractioni $n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-m+1}{m}$. Q. E. D.

COR. I.

Biquadratica ($x^4 * + qx^2 - rx + s = o$) cujus ſecundus terminus deeſt, reduci poteſt in quadraticam ($x^2 - Px + Q = o$,) æquationis ($v^6 - av^4 + bv^2 - c = o$) ſex dimenſionum ope, cujus formula eſt cubica.

COR. II.

Hac vero methodo æquatio reducens plures habet dimenſiones quam data æquatio; fruſtra eſſet igitur, per talem reductionem æquationum reſolutionem generalem quærere. *

† *De Moiv.* Miſc. et Anal. p. 180. * *Newt.* Arith. Univ. p. 237.

PROB.

PROB. VI.

Nonnullas æquationes reducere, eas in duas æquales partes ita dividendo, ut ex utrâque parte eadem radix extrahatur.

EXEMPLUM I.

Biquadraticam æquationem $x^4+2px^3=qx^2+rx+s$ in quadraticam reducere.

Addatur ad utramque æquationis partem

$$\overline{p^2+2n}\times x^2+2pnx+n^2 \text{ \& prodit}$$

$$x^4+2px^3+\overline{p^2+2n}\times x^2+2pnx+n^2=\overline{p^2+2n+q}\times x^2+\overline{2pn+r}\times x+\overline{s+n^2}.$$

Supponatur $4\times\overline{s+n^2}\times\overline{p^2+2n+q}=\overline{2pn+r}^2$ reducatur hæc æquatio & tranſponantur termini, & fit cubica

$$8n^3+4qn^2+\overline{8s-4rp}\times n+4qs+4p^2s-r^2=0.$$

Hinc inveniatur incognita quantitas (n), quâ pro ſuo valore (n) in æquatione

$$x^2+2px^3+\overline{p^2+2n}\times x^2+2pnx+n^2=\overline{p^2+2n+q}\times x^2+\overline{2pn+r}\times x+s+n^2$$

ſubſtitutâ, & extractâ utrobique radice quadraticâ, reſultat æquatio quadratica

$$x^2+px+n=\sqrt{\overline{p^2+2n+q}}\times x+\sqrt{\overline{s+n^2}}.$$

Ex. Gr. Sit biquadratica æquatio $x^4-25x^2+60x-36=0$, pro p, q, r, s, ſcribantur $0, 25, -60, 36$ reſpective in reducenti æquatione

$$8n^3+4qn^2+\overline{8s-4rp}\times n+4qs+4p^2s-r^2=0 \text{ \& fit}$$

$8n^3+100n^2+288n=0$, cujus radices ſunt reſpective $0, -8, -\frac{9}{2}$, quibus pro ſuo valore (n) in æquatione

$$x^2+2px+n=\sqrt{\overline{q+p^2+2n}}\times x+\sqrt{\overline{s+n^2}}$$

ſubſtitutis, reſultant tres æquationes

$$x^2 = \sqrt{25}\,x+\sqrt{36}$$

$$x^2-8=\sqrt{9}\,x+\sqrt{100}$$

$$x^2-\tfrac{9}{2}=\sqrt{16}\,x+\sqrt{\tfrac{225}{4}}$$

Ex ſingulis his tribus æquationibus, radices vel ſunt $1, 2, 3, -6$ (qui numeri ſunt radices biquadraticæ æquationis $x^4-25x^2+60x-36=0$) vel earum negativæ $-1, -2, -3, +6$.

EXEMPLUM II.

Æquationem quadrato-cubicam ($x^5+qx^3+rx^2+sx+t=0$) in cubicam reducere.

Multiplicetur data æquatio per (x) & reſultat

$$x^6+qx^4+rx^3+sx^2+tx=0,$$

addatur ad utramque æquationis partem $\overline{kx^3+dx+b}^2$, & prodit

$$\overline{kx^3+dx+b}^2=\overline{k^2+1}x^6+\overline{2kd+q}x^4+\overline{2kb+r}x^3+\overline{d^2+s}x^2+\overline{2db+t}x+b^2.$$

Supponantur $2\sqrt{k^2+1}\times b=2kb+r$, $2\sqrt{k^2+1}\times\sqrt{d^2+s}=2kd+q$, $2\sqrt{d^2+s}\times b=2db+t$ quibus reductis reſultant tres æquationes

$$4b^2=4rkb+r^2$$

$$4sk^2+4d^2+4s=4qkd+q^2$$

$$4sb^2=4tdb+t^2$$ e primâ æquatione ſequitur $k=\frac{4b^2-r^2}{4rb}$, e tertiâ vero $d=\frac{4sb^2-t^2}{4tb}$, quibus quantitatibus pro ſuis valoribus in ſecundâ æquatione ſubſtitutis, reſultat æquatio biquadratica, formulâ vero quadraticæ

$$4\times\frac{\overline{4sb^2-t^2}^2}{\overline{4tb}^2}+4s\times\frac{\overline{4b^2-r^2}^2}{\overline{4rb}^2}+4s=4q\times\frac{4b^2-r^2}{4rb}\times\frac{4sb^2-t^2}{4tb}+q^2$$

Reducatur hæc æquatio & fit

$$16sb^4+4qrtb^2+r^2t^2=0.$$

Ex hâc æquatione inveniri poteſt incognita quantitas (b) & conſequenter $k=\frac{4b^2-r^2}{4rb}$ & $d=\frac{4sb^2-t^2}{4tb}$, quibus quantitatibus pro ſuis valoribus ſubſtitutis, reſultat cubica æquatio

$$Ax^3+Bx+C=0.$$

Ex. Gr. Sit data quadrato-cubica æquatio,

$$x^5+35x^3+30x_2+304x-480=0.$$

Pro (q,r,s,t) ſcribantur in reducenti æquatione reſpective 35, 30, 304, —480, & fit

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Note sur la page 36. de l'ouvrage de Waring intitulé. Miscellanea analytica.

L'équation du 5.ème degré $x^5+qx^3+rx^2+sx+t=0$ est, par sa forme, telle que en nommant a, a', a'', a''', a^{IV} les cinq racines, on doit avoir leurs somme

$$(1) \ldots a+a'+a''+a'''+a^{IV}=0$$

puisque le second terme manque dans l'équation. Si l'on admet que l'équation proposée a deux racines égales et de signe contraire il sera facile de ramener la solution de l'équation du cinquième degré à une équation du troisième degré. Car en posant $a+a'=0$, et supposant $x^5+qx^3+rx^2+sx+t=(x^2-a^2)(x^3+mx+n)$ on trouvera aisément, que le second membre devient identique avec le premier, en faisant

$$x^5+qx^3+rx^2+sx+t=\left(x^2+\frac{t}{r}\right)\left(x^3+\frac{sr}{t}x+r\right)$$

pourvu qu'on ait $\frac{s.r}{t}+\frac{t}{r}=q$, ou bien

$$(2) \ldots sr^2+t^2=qrt.$$

Ainsi cette équation exprime la condition nécessaire pour que la proposée du cinquième degré ait deux racines égales et de signe contraire. Lorsque elle sera satisfaite on aura

$$(3) \ldots x=\pm\sqrt{-\frac{t}{r}},$$

$$(4) \ldots x^3+\frac{sr}{t}x+r=0.$$

L'exemple de Waring doit être écrit ainsi

$$(5) \ldots x^5-35.x^3+30x^2+304.x-480=0.$$

Car en laissant le signe + au coefficient 35 il

serait impossible de satisfaire à l'équation

$$x^5+35x^3+30x^2+304x-480=0$$

par les racines 2, 3, −5.

Dans le cas de l'équation (5) nous avons

$q=-35$; $r=30$; $s=304$; $t=-480$.

Donc la formule (3) donne

$$x=\pm\sqrt{\frac{480}{30}}=\pm\sqrt{16}=\pm 4.$$

La formule (4) devient

$$x^3-\frac{304.30}{480}x+30=0$$

ou bien $x^3-19x+30=0$.

Or il est facile de trouver que 2, 3, −5 sont les racines de cette équation.

Telle est la solution la plus simple de ce problème de Waring: La méthode de l'auteur est plus compliquée: il est conduit à l'équation

(A)·· $(Kx^2+dx+b)^2=(\pm x^2\sqrt{K^2+t}\pm x\sqrt{d^2+s}\pm b)^2$

où l'on a;

(6).. $K=\frac{4b^2-r^2}{4rb}$, (7).. $d=\frac{4s.b^2-t^2}{4tb}$; et

(8).. $4sK^2+4d^2+4s=4.qKd+q^2$.

En substituant dans l'équation (8) les valeurs de K et d fournies par les équations (6) et (7) on obtient

(9)... $(sr^2+t^2-qrt)(16.sb^4+4qrt.b^2+r^2t^2)=0$

Ainsi cette équation peut avoir lieu de deux manières: ou en admettant que les coefficients q, r, s, t soient tels qu'on a

(10)... $sr^2+t^2-qrt=0$,

ou en déterminant b^2 de manière qu'on ait

(11).. $16.sb^4+4qrt.b^2+r^2t^2=0$.

Waring ne dit rien sur l'équation (10), et il adopte l'équation (11). Mais Waring n'a pas même remarqué qu'on peut extraire les deux racines $\sqrt{k^2+1}$, $\sqrt{c^2+s}$. En effet nous avons

$$k^2+1=\frac{(4b^2-r^2)^2+16.r^2b^2}{16r^2b^2}=\left(\frac{4b^2+r^2}{4rb}\right)^2 ;$$ partant

$$\sqrt{k^2+1}=\frac{4b^2+r^2}{4rb}.$$ Ensuite on a

$$c^2+s=\frac{(4sb^2-t^2)^2+16.st^2b^2}{16t^2b^2}=\left(\frac{4.sb^2+t^2}{4tb}\right)^2.$$

$$\sqrt{c^2+s}=\frac{4sb^2+t^2}{4tb}$$

Il suit delà que l'équation (a) revient à celle-ci;

$$x^3\left(\frac{4b^2-r^2}{4rb}\right)+x\left(\frac{4sb^2-t^2}{4tb}\right)+b$$
$$=\pm\left\{\pm x^3\left(\frac{4b^2+r^2}{4rb}\right)\pm x\left(\frac{4sb^2+t^2}{4tb}\right)\pm b\right\}.$$

De là les deux combinaisons

$$x^3\left(\frac{4b^2-r^2}{4rb}\right)+x\left(\frac{4sb^2-t^2}{4tb}\right)+b=-x^3\left(\frac{4b^2+r^2}{4rb}\right)+x\left(\frac{4sb^2+t^2}{4tb}\right)+b;$$

$$x^3\left(\frac{4b^2-r^2}{4rb}\right)+x\left(\frac{4sb^2-t^2}{4tb}\right)+b=x^3\left(\frac{4b^2+r^2}{4rb}\right)-x\left(\frac{4sb^2+t^2}{4tb}\right)-b;$$

La première de ces équations donne

(12).. $x=\pm\sqrt{\frac{rt}{4b^2}}$

et la seconde est réductible à celle-ci;

(13).. $x^3-\frac{4b^2s}{rt}.x-\frac{4b^2}{r}=0$.

Ainsi la solution de Waring est comprise dans les trois équations (11), (12) et (13). Dans l'exemple

$$x^5-35.x^3+30x^2+304.x-480=0$$

l'équation (11) donne $b^2=(15)^2\left(-\frac{35\pm3}{38}\right)$

En prenant $b^2 = -(15)^2$, la formule (12) donnera $x = \pm 4$ et l'équation (13) deviendra, comme par la méthode précédente $x^2 - 19x + 30 = 0$.

Au reste l'équation (11) donne

$$-b^2 = -\frac{rt}{8.s}\left\{-q \pm \sqrt{q^2-4s}\right\}$$

en prenant seulement le signe $+$ on obtient

$$\frac{rt}{4b^2} = +\frac{1}{2}\cdot\left(-q+\sqrt{q^2-4s}\right):$$

Donc les formules (12) et (13) donnent

$$(14)\ldots\quad x = \pm\sqrt{\frac{-q+\sqrt{q^2-4s}}{2}};$$

$$(15)\ldots\quad x^3 + \frac{2x.s}{q-\sqrt{q^2-4s}} + \frac{2t}{q-\sqrt{q^2-4s}} = 0.$$

La formule (14) exige deux racines égales et de signe contraire; ce qui entraîne l'équation (2); partant nous avons $q = \frac{sr^2+t^2}{rt}$ et $\sqrt{q^2-4s} = \frac{sr^2-t^2}{rt}$; d'où l'on tire

$$\frac{-q+\sqrt{q^2-4s}}{2} = -\frac{t}{r}$$: alors les équations (14) et (15) donnent

$$x = \pm\sqrt{\frac{-t}{r}};$$

$$x^3 + x.\frac{sr}{t} + r = 0.$$

ce qui revient au résultat de la 1ère méthode.

Je pense que ~~le signe abrégé~~ Waring voulait dire h:c c'est-à-dire hoc est et non q:c ~~tient lieu de la phrase~~ ~~quod eodem redit~~ de sorte que Waring a voulu écrire "Haec regula fallit, si summa duarum, vel ~~quod~~ hoc est ~~eadem~~ ~~redit~~, si summa trium radicum nihilo sit aequalis".

Effectivement après avoir écrit l'équation du 5e degré sans second terme on ne peut dire que la somme de deux racines est nulle sans dire que la somme des trois autres est aussi nulle.

Turin le 23. 8bre 1834.

J. Plana

Note sur la page (37).

Soit l'équation du sixième degré

$$(1) \ldots x^6+px^4+qx^3+rx^2+sx+t=0.$$

En ajoutant de part et d'autre le carré $(kx^3+dx+b)^2$ on obtient

$$(kx^3+dx+b)^2 = (k^2+1)x^6+(d^2+r)x^2+(t+b^2)$$
$$+(2kd+p)x^4+(2kb+q)x^3+(s+2bd)x.$$

Mais si l'on suppose

$$2\sqrt{k^2+1}\cdot\sqrt{d^2+r} = 2kd+p; \quad 2\sqrt{k^2+1}\cdot\sqrt{t+b^2} = 2kb+q;$$
$$2\sqrt{d^2+r}\cdot\sqrt{t+b^2} = s+2bd;$$

l'équation précédente sera réductible à celle-ci;

$$(kx^3+dx+b)^2 = \left(\pm x^3\sqrt{k^2+1} \pm x\sqrt{d^2+r} \pm \sqrt{t+b^2}\right)^2$$

d'où l'on tire ces deux équations

$$kx^3+dx+b = -x^3\sqrt{k^2+1}+x\sqrt{d^2+r}+\sqrt{t+b^2}$$
$$kx^3+dx+b = x^3\sqrt{k^2+1}-x\sqrt{d^2+r}-\sqrt{t+b^2}$$

ou bien

$$(2) \ldots x^3+x\left\{\frac{d-\sqrt{d^2+r}}{k+\sqrt{k^2+1}}\right\}+\frac{b-\sqrt{t+b^2}}{k+\sqrt{k^2+1}}=0$$

$$(3) \ldots x^3+x\left\{\frac{d+\sqrt{d^2+r}}{k-\sqrt{k^2+1}}\right\}+\frac{b+\sqrt{t+b^2}}{k-\sqrt{k^2+1}}=0.$$

Ainsi l'équation (1) serait le produit des équations (2) et (3). Les équations qui déterminent k, d, b sont

$$k^2\cdot 4t - k\cdot 4qb = q^2-4t-4b^2 \ldots (4)$$ (36)
$$d^2\cdot 4t - d\cdot 4sb = s^2-4rb^2-4tr \ldots (5)$$
$$k^2\cdot 4r+4d^2+4r-4pkd-p^2=0 \ldots (6).$$

L'équation (4) donne

$$K = \frac{qb + \sqrt{(q^2-4t)t - b^2(4t-q^2)}}{2t};$$

L'équation (5) donne

$$d = \frac{sb + \sqrt{(s^2-4tr)t - b^2(4rt-s^2)}}{2t}.$$

En multipliant l'équation (6) par t on a

$$4tK^2r + 4d^2t = t(p^2 + 4pKd - 4r):$$

en éliminant de cette équation $4K^2t$ et $4d^2t$ au moyen des équations (4) et (5) il vient

$$r\{K4qb + q^2 - 4t - 4b^2\} + \{d.4sb + s^2 - 4rb^2 - 4tr\}$$
$$= t\{p^2 + 4pKd - 4r\}$$

ou bien ;

~~$K\{4qbr - 4pt$~~

$$K.4qbr + d.4bs - Kd.4pt + r(q^2 - 4t - 4b^2)$$
$$+ (s^2 - 4rb^2 - 4tr) - t(p^2 - 4r) = 0$$

ou bien

(7). $0 = K.4qbr + d.4bs - Kd.4pt + rq^2 - tp^2 + s^2 - 4tr - 8r$

En substituant ici les valeurs de K et d et faisant ensuite disparaître les radicaux on doit trouver l'équation du quatrième degré par rapport à b^2 dont parle Waring. Mais par cette méthode on n'a pas explicitement les conditions entre les coefficiens de l'équation du sixième degré qui sont nécessaires pour la

possibilité d'une telle décomposition. Pour éclaircir cette méthode il faudra appliquer à ce cas la méthode exposée dans la Résolution des équations numériques de Lagrange (page 196).

Soit $k = \frac{qb+\sqrt{M}}{2t}$ $d = \frac{sb+\sqrt{H}}{2t}$; nous aurons

$$kd = \frac{1}{4t^2}\left\{ qsb^2 + qb\sqrt{H} + sb\sqrt{M} + \sqrt{M\cdot H} \right\}$$

substituant ces valeurs dans l'équation (>) on obtient

$$\frac{2qbr}{t}(qb+\sqrt{M}) + \frac{2bs}{t}(sb+\sqrt{H}) - \frac{p}{t}\left\{qsb^2+qb\sqrt{H}+sb\sqrt{M}+\sqrt{MH}\right\}$$

$$+ rq^2 - tp^2 + s^2 - 4tr - 8rb^2 = 0$$

ou bien

$$\left\{\begin{array}{l} b^2\left(\frac{2q^2r}{t} + \frac{2s^2}{t} - \frac{pqs}{t} - 8r\right) + rq^2 - tp^2 + s^2 - 4tr \\ + \sqrt{M}\left(\frac{2qbr}{t} - \frac{psb}{t}\right) + \sqrt{H}\left(\frac{2bs}{t} - \frac{pqb}{t}\right) - \frac{p}{t}\sqrt{MH} \end{array}\right.$$

ou bien

$$\left\{\begin{array}{l} b^2(2q^2r + 2s^2 - pqs - 8rt) + rq^2t - t^2p^2 + ts^2 - 4rt^2 \\ + b\sqrt{M}(2qr - ps) + b\sqrt{H}(2s - pq) - p\sqrt{MH} \end{array}\right.$$

Soit $(q^2-4t)t = f$; $4t - q^2 = g$.

$(s^2-4tr)t = Q$ $4rt - s^2 = \beta$.

$\sqrt{M} = \sqrt{f - gb^2}$ $\sqrt{H} = \sqrt{Q - \beta . b^2}$

l'équation précédente devient de cette forme ;

$$A + b^2B + bC\sqrt{f-gb^2} + bD\sqrt{Q-\beta b^2} = p\sqrt{(f-gb^2)(Q-\beta b^2)}$$

en élevant au carré

$$(A+b^2B)^2 + b^2C^2(f-gb^2) + b^2D^2(Q-\beta b^2)$$

$$A + b^2B = p\sqrt{(f-gb^2)(Q-\beta b^2)} - bC\sqrt{f-gb^2} - bD\sqrt{Q-\beta b^2}$$

$$A+b^2B-p\sqrt{(f-gb^2)(\alpha-\beta b^2)} = -bC\sqrt{f-gb^2}-bD\sqrt{\alpha-\beta b^2}$$

en faisant le carré il viendra

$$(A+b^2B)^2+p^2(f-gb^2)(\alpha-\beta b^2)-2p(A+b^2B)\sqrt{(f-gb^2)(\alpha-\beta b^2)}$$
$$= b^2C^2(f-gb^2)+b^2D^2(\alpha-\beta b^2)$$
$$+b^2CD\sqrt{(f-gb^2)(\alpha-\beta b^2)}$$

ou bien

$$(A+b^2B)^2+p^2(f-gb^2)(\alpha-\beta b^2)-b^2C^2(f-gb^2)$$
$$-b^2D^2(\alpha-\beta b^2) = \left[b^2CD+2p(A+b^2B)\right]\sqrt{(f-gb^2)(\alpha-\beta b^2)}$$

en carrant les deux membres de cette équation il viendra

$$\left\{\begin{array}{l}(A+b^2B)^2+p^2(f-gb^2)(\alpha-\beta b^2)\\ -b^2C^2(f-gb^2)-b^2D(\alpha-\beta b^2)\end{array}\right\}^2$$
$$= \left[b^2CD+2p(A+b^2B)\right]^2(f-gb^2)(\alpha-\beta b^2).$$

C'est l'équation du huitième degré par rapport à b.

$$16\times 304b^4 - 4\times 35\times 30\times 480b^2 + 30^2\times 480^2 = 0:$$

dividatur hæc æquatio per (16×16) & reſultat

$$19b^4 - 35\times 15^2 b^2 + 30^4 = 0:$$

hujuſce æquationis radices erunt $b = \pm\dfrac{\sqrt[2]{\pm 675 + 7875}}{\sqrt{38}}$, ſubſtituatur hæc quantitas pro ejus valore in æquationibus

$$k = \frac{4b^2 - r^2}{4rb} \ \&\ d = \frac{4sb^2 - t^2}{4tb},$$

& reſultant quantitates

$$k = \frac{\pm 45 - 45}{2\sqrt{38}\sqrt{\pm 675 + 7875}}$$

$$d = \frac{\pm 855 - 855}{2\sqrt{38}\sqrt{\pm 675 + 7875}}$$

ſubſtituantur hæ quantitates pro ſuis valoribus in cubicâ æquatione,

$$kx^3 + dx + b = \pm\sqrt{k^2+1}\,x^3 \pm \sqrt{d^2+s}\,x \pm q$$

& reſultant ex ſingulâ ſubſtitutione æquationes, quarum ſex radices erunt 0, 2, 3, 4, —4, —5 datæ æquationis radices.

EXEMPLUM III.

Æquationem cubo-cubicam $(x^6 * + px^4 + qx^3 + rx^2 + sx + t = 0)$ in cubicam reducere.

Addatur ad utramque æquationis partem $\overline{kx^3 + dx + b}^2$ & prodit $\overline{ka^3 + dx + b}^2 = \overline{k^2+1}\times x^6 + \overline{2kd+q}\times x^4 + \overline{2kb+r}\times x^3 + \overline{d^2+s}\times x^2 + \overline{2dq+t}\times x + b^2 + v.$

Supponantur $2\sqrt{k^2+1}\times\sqrt{r+d^2} = p + 2kd,$ $2\sqrt{k^2+1}\times\sqrt{t+b^2} = 2kb + q$ & $2\sqrt{r+d^2}\times\sqrt{t+b^2} = s + 2db$ quibus reductis reſultant tres æquationes

$$4tk^2 + 4b^2 + 4t = 4qkb + q^2$$
$$4rk^2 + 4d^2 + 4r = 4pkd + p^2$$
$$4tr + 4td^2 + 4rb^2 = 4sdb + s^2$$

e primâ æquatione ſequitur $k = \sqrt{\dfrac{q^2 - 4t}{4t} - \dfrac{4t - q^2}{4t^2}b^2} + \dfrac{qb}{2t}$

e tertiâ

e tertiâ $d=\sqrt{\frac{s^2-4tr}{4t}-\frac{4rt-s^2}{4t^2}\times b^2}+\frac{sb}{2t}$ quibus quantitatibus in ſecundâ æquatione pro ſuis valoribus ſubſtitutis, prodit

$$4r\times\overline{\sqrt{\frac{q^2-4t}{4t}-\frac{4t-q^2}{4t^2}\times b^2}+\frac{qb}{2t}}^{\,2}+4\times\overline{\sqrt{\frac{s^2-4tr}{4t}-\frac{4rt-s^2}{4t^2}\times b^2}+\frac{sb}{2t}}^{\,2}$$

$$+4r=4p\times\overline{\sqrt{\frac{q^2-4t}{4t}-\frac{4t-q^2}{4t^2}b^2}+\frac{qb}{2t}}\times\overline{\sqrt{\frac{s^2-4tr}{4t}-\frac{4rt-s^2}{4t^2}b^2}+\frac{sb}{2t}}$$

$+p^2$.

Reducatur hæc æquatio, ita ut exterminentur ſurdæ quantitates & reſultat æquatio, quæ octo habet dimenſiones, ſed ejus formula eſt biquadratica.

Ex hâc æquatione inveniri poteſt ejus incognita quantitas (b) & conſequenter $\sqrt{\frac{q^2-4t}{4t}-\frac{4t-q^2}{4t^2}b^2}+\frac{qb}{2t}=k$ &

$\sqrt{\frac{s^2-4tr}{4t}-\frac{4rt-s^2}{4t^2}b^2}+\frac{sb}{2t}=d$ quibus quantitatibus in æquatione

$$\overline{kx^3+dx+b}^{\,2}=\overline{k^2+1}\,x^6+\overline{2kd+p}\,x^4+\overline{2kb+q}\,x^3+\overline{d^2+r}\times x^2+\overline{2db+s}\times x+b^2+t$$

pro ſuis valoribus (b, k, d) ſubſtitutis, & æquationis reſultantis extracta utrobique radice; prodit cubica æquatio

$$kx^3+dx+b=\sqrt{k^2+1}\,x^3+\sqrt{d^2+r}\,x+\sqrt{b^2+t}.$$

Infinitæ ſunt conſimiles æquationes reducendi methodi; ſed facile patet ex hoc fonte haud generalem æquationum reductionem oriri poſſe.

PROB. VII.

Nonnullas æquationes reſolvere, eas transformando in alias, quarum radices ad radices datæ æquationis datam habent relationem.

EXEMPLUM I.

Sit æquatio $y^3+Ay-B=o$, aſſumatur æquatio $v-\frac{A}{3v}=y$, quâ quantitate pro y ſubſtitutâ, transformatur data æquatio in æquationem $v^6-Bv^3-\frac{A^3}{27}=o$, cujus ope invenitur v & conſequenter $v-\frac{A}{3v}=y$.

EX-

EXEMPLUM. II.

Sit æquatio $y^4 + py^2 + qy + r = o$. aſſumatur æquatio $y^2 + ay + b = v$, reducantur hæ duæ æquationes in unam, ita ut exterminetur y, & fit æquatio reſultans

$$\begin{array}{llll} v^4 - 4b\,v^3 & +6b^2\;\;v^2 & +2pr\;\;\;v & \\ +2p & +2r & -q^2 & \\ & +3aq & +apq & \\ & -6pb & +4a^2r & \\ & +p^2 & -4br & \\ & +a^2p & -2bp^2 & \\ & & +a^3q & \\ & & -6abq & \\ & & +6b^2p & \\ & & -2a^2bp & \\ & & -4b^3 & \end{array} + \left|\begin{array}{r} \overline{\tfrac{1}{2}a^2 - b}^{\,2} + a^2.\overline{\tfrac{1}{4}a^2 - b} \\ +p\,\overline{\tfrac{1}{2}a^2 - b} \\ -\tfrac{1}{2}.qa \\ r \end{array}\right|^2 - \overline{\tfrac{1}{4}a^2 - b}.\left|\begin{array}{r} 2a.\overline{\tfrac{1}{2}a^2 - b} \\ -pa \\ +q \end{array}\right|^2 = o$$

nihilo æqualis fiat ſecundus æquationis terminus $\left(\begin{array}{l} -4b \times v^3 \\ +2p \end{array}\right)$ & fit $b = \frac{p}{2}$, qua quantitate pro ſuo valore in quarto termino ſubſtituta, erit

$$\begin{array}{l} qa^3 - p^2a^2 - 2pqa - q^2 \\ +4r \end{array}$$ quod nihilo fiat æquale & reſultat cubica, cujus ope invenitur $(a,)$ in æquatione reſultanti pro a & b ſubſtituantur earum valores, & reſultat æquatio biquadratica $(v^4 - Pv^2 + Q = o)$ formulâ vero quadraticæ, cujus ope invenitur (v) & conſequenter ex aſſumptâ æquatione $y^2 + ay + b = v$ inveniri poteſt datæ æquationis radix (y).

COROLLARIUM.

Sit data æquatio

$$x^n - px^{n-1} + qx^{n-2}, \text{\&c.} = o.$$

aſſumatur $ax^2 + bx + c = v$, & transformentur hæ duæ æquationes in unam, cujus radix eſt (v); tum ita aſſumi poſſunt coefficientes a, b, c, ut quilibet duo reſultantis æquationis termini nihilo æquales fiant.

Et ſic de transformandis æquationibus, ita ut plures termini e reſultanti æquatione tollantur.

Sed

Sed haud tolli poſſunt e reſultanti æquatione generaliter plures termini, quam incognitæ ſunt in aſſumptâ quantitate coefficientes.

COROLLARIUM.

In æquationibus multarum dimenſionum, reſolvens plures habet quam data æquatio dimenſiones: fruſtra eſſet igitur per talem methodum generalem æquationum reſolutionem quærere.

Sint incognitæ in aſſumptâ quantitate coefficientes plures, quam termini tollendi.

Inveniantur termini tollendi, nihilo reſpective fiant æquales, & transformentur reſultantes æquationes in unam ita ut exterminentur quædam incognitæ coefficientes, & reſultat æquatio duas vel plures incognitas quantitates habens; inveniantur ex ſingulis his integræ vel rationales radices ſibi ipſis correſpondentes, quibus ſubſtitutis, tolluntur termini requiſiti.

COROLLARIUM.

Per hoc problema nonnullæ æquationes $n\times\frac{n-1}{2}$, $n\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. dimenſionum deprimi poſſunt in æquationes (n) dimenſionum.

Aſſumatur æquatio (n) dimenſionum, & per problema primum inveniantur aliæ, quarum radices ſunt ſummæ ex quibuſque duabus, tribus, &c. & earum dimenſiones erunt $n\times\frac{n-1}{2}$, $n\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. hæ æquationes reſultantes iterum deprimi poſſunt in (n) dimenſionum æquationem.

Et ſic inveniri poſſunt innumeræ æquationes, quas deprimere licet.

LEMMA.

Æquatio

$$x^{2n}\pm x^{2n-1}+x^{2n-2}\pm x^{2n-3}+x^{2n-4}\pm \&c.=0,$$

in quadraticam $x^2+vx+1=0$ reduci poteſt æquationis ope

$$v^n\mp v^{n-1}-\overline{n-1}\,v^{n-2}\pm\overline{n-2}\,v^{n-3}+\overline{n-2}\times\frac{\overline{n-3}}{2}v^{n-4}\mp\overline{n-3}\times\frac{\overline{n-4}}{2}v^{n-5}-\overline{n-3}\times\frac{\overline{n-4}}{2}\times\frac{n-5}{3}v^{n-6}\pm\overline{n-4}\times\frac{\overline{n-5}}{2}\times\frac{\overline{n-6}}{3}\times v^{n-7}+\overline{n-4}\times\frac{\overline{n-5}}{2}\times\frac{\overline{n-6}}{3}\times\frac{\overline{n-7}}{4}v^{n-8}\mp \&c.=0.$$

COROLLARIUM.

Quatuor radices æquationis ($x^4+1=0$) erunt $\pm\sqrt{\tfrac{1}{2}}\pm\sqrt{-\tfrac{1}{2}}$, quinque vero radices æquationis ($x^5+1=0$) ſunt -1, $\dfrac{\pm\sqrt{1\frac{1}{4}}+\frac{1}{2}}{2}\pm\dfrac{\sqrt{-2\frac{1}{2}\pm\sqrt{1\frac{1}{4}}}}{2}$, ſex vero radices æquationis ($x^6+1=0$) ſunt

$$\pm\sqrt{-1},\ \frac{\pm\sqrt{3}\pm\sqrt{-1}}{2}$$

LEMMA V.

Datâ quantitate utcunque irrationali, invenire alteram, quæ in datam ducta, rationale productum facit.

Datæ irrationalis quantitatis inveniantur ſinguli diverſi valores, quibus in ſeſe continuo ductis, contentum erit rationale productum; dividatur hoc productum per datam quantitatem, & quotiens erit irrationalis quantitas quæſita.

EXEMPLUM.

Invenire irrationalem quantitatem, quæ in $\sqrt[2]{a+bx}+\sqrt[3]{c+dx}$ ducta, rationale productum facit.

Singuli diverſi valores datæ irrationalis quantitatis $\sqrt[2]{a+bx}+\sqrt[3]{c+dx}$ erunt reſpective

$$\sqrt{a+bx}+\sqrt[3]{c+dx},\ -\sqrt{a+bx}+\sqrt[3]{c+dx},\ \sqrt{a+bx}-\frac{1+\sqrt{-3}}{2}\sqrt[3]{c+dx},\ -\sqrt{a+bx}-\frac{1+\sqrt{-3}}{2}\sqrt[3]{c+dx},\ \sqrt{a+bx}-\frac{1-\sqrt{-3}}{2}\sqrt[3]{c+dx},\ -\sqrt{a+bx}-\frac{1-\sqrt{-3}}{2}\times\sqrt[3]{c+dx}.$$

Contentum $(\overline{a+bx}^{3}-\overline{c+dx}^{2})$ ex his omnibus in ſeſe ductis erit rationale productum, & contentum

$$(\overline{a+bx}^{\frac{5}{2}}-\overline{a+bx}^{\frac{4}{2}}\times\overline{c+dx}^{\frac{1}{3}}+\overline{a+bx}^{\frac{3}{2}}\times\overline{c+dx}^{\frac{2}{3}}-\overline{a+bx}^{\frac{2}{2}}\times\overline{c+dx}^{\frac{3}{3}}+\overline{a+bx}^{\frac{1}{2}}\times\overline{c+dx}^{\frac{4}{3}}-\overline{c+dx}^{\frac{5}{3}})$$

ex omnibus niſi prima (quæ ſit data) in ſeſe ductis, erit irrationalis quantitas quæſita.

COR. I.

Hinc colligi poteſt, e quâ multiplicatione vel conſtructione orientur quæcunque datæ quantitates.

COR. II.

Datâ æquatione ſurdas involvente, ex hoc lemmate manifeſto conſtant dimenſiones, ad quas reſultans æquatio aſſurgit.

COR. III.

Hoc lemmate irrationalis quantitas (A) a denominatore in numeratorem & vice verſâ, transferri poſſit.

Inveniatur rationalis quantitas (B), cujus data irrationalis eſt diviſor, ſit vero $C = \frac{B}{A}$; & pro A ſubſtituatur $\frac{B}{C}$, & fit.

COR. IV.

Quantitate datâ, quæ ſimplicium terminorum juxta dimenſiones cujuſlibet literæ (x) progredientium ſeriem ($a+b+c+d+e+f+g$, &c.) exprimit, invenire quantitatem, alternis ſeriei terminis ($a+c+e+g$, &c.) æqualem, vel terminis ſeriei propoſitæ binis vel ternis vel quaternis vel denique (n) intervallis a ſe invicem diſtantibus.

Sint $\alpha, \beta, \gamma, \delta$, &c. reſpectivæ radices ($n+1$) poteſtatis numeri (1).

In datâ quantitate pro literâ (x) ſubſtituantur reſpective $\alpha x, \beta x, \gamma x, \delta x$, &c. quantitatum reſultantium ſumma per numerum ($n+1$) diviſa, erit quæſita quantitas.

EXEMPLUM. I.

Sit quantitas $\overline{a+x}^m$, quæ exprimit ſeriem ſequentem ſimplicium terminorum juxta dimenſiones literæ (x) progredientium,

$$a^m + m \cdot a^{m-1} x + m \frac{m-1}{2} a^{m-2} x^2 + \text{\&c.}$$

invenire quantitatem alternis ſeriei terminis æqualem.

Pro $n+1$ ſcribatur numerus (2,) & quoniam radices quadraticæ unitatis ſunt 1 & -1, pro (x) in data quantitate ſcribantur reſpective $+x$ & $-x$; & reſultant quantitates $\overline{a+x}^m$ & $\overline{a-x}^m$, quarum ſumma per numerum ($n+1=2$) diviſa, erit quæſita quantitas

$$i.e.\ \frac{\overline{a+b}^m + \overline{a-b}^m}{2} = a^m + m \times \frac{m-1}{2} a^{m-2} b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 + \text{\&c.}$$

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EXEMPLUM II.

Eâdem quantitate datâ, ſit vero ($n+1=3$;) cubicæ radices numeri (1) ſunt 1, $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$; in datâ igitur quantitate pro x ſubſtituantur x, $\frac{-1+\sqrt{-3}}{2}x$, $\frac{-1-\sqrt{-3}}{2}x$, reſpective, & reſultant quantitates $\overline{a+x}^m$, $\overline{a-\frac{1+\sqrt{-3}}{2}x}^m$, $\overline{a-\frac{1-\sqrt{-3}}{2}x}^m$, quarum ſumma per numerum ($n+1=3$) diviſa, erit quæſita quantitas *i.e.* $\frac{\overline{a+x}^m+\overline{a-\frac{1+\sqrt{-3}}{2}x}^m+\overline{a-\frac{1-\sqrt{-3}}{2}x}^m}{3}=$

$$=a^m+m\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}x^3+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{n-4}{5}\times\frac{m-5}{6}\times a^{m-6}x^6 \text{ \&c.}$$

EXEMPLUM III.

Datâ quantitate $\overline{a+bx+\overline{c+dx+ex^2}^l}^m$ exprimente ſeriem terminorum juxta dimenſiones literæ (x) progredientium, invenire quantitatem alternis hujus ſeriei terminis æqualem. Hoc caſu $\overline{n+1}=2$, ergo pro x in data quantitate ſcribantur reſpective x & $-x$; & reſultant quantitates $\overline{a+bx+\overline{c+dx+ex^2}^l}^m$ & $\overline{a-bx+\overline{c-dx+ex^2}^l}^m$: quarum ſumma per numerum ($n+1=2$) diviſa, eſt quæſita quantitas

$$\left(\frac{\overline{a+bx+\overline{c+dx+ex^2}^l}^m+\overline{a-bx+\overline{c-dx+ex^2}^l}^m}{2}.\right)$$

PROB. VIII.

Datâ reſolutione, vel reductione, æquationem invenire.

Præcedentis lemmatis vel problematis primi ope, ita reducatur data æquatio, ut ex eâ exterminentur irrationales quantitates, & ſit problema.

Exemplum I.

Sit (x) incognita quantitas, & data reſolutio

$x = \sqrt[3]{-\tfrac{1}{2}b + \sqrt{\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3}} + \sqrt[3]{-\tfrac{1}{2}b - \sqrt{\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3}}$; ita reducatur data æquátio, ut exterminentur irrationales quantitates, & reſultat æquatio $x^9 + 3bx^6 + \overline{3b^2 + a}^3 x^3 + b^3 = 0$, cujus, cubicæ æquationes $x^3 + ax + b = 0$, $x^3 - \frac{1 - \sqrt{-3}}{2} ax + b = 0$, $x^3 - \frac{1 + \sqrt{-3}}{2} ax + b = 0$, ſunt diviſores.

Exemplum II.

Sit reſolutio $x = \sqrt[3]{-\tfrac{1}{2}b + \sqrt{\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3}} - \dfrac{\tfrac{1}{3}a}{\sqrt[3]{-\tfrac{1}{2}b + \sqrt{\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3}}}$; exterminentur irrationales quantitates, & æquatio reſultans erit $x^6 + 2ax^4 + bx^3 + a^2x^2 + abx = 0$, cujus, æquationes cubicæ $x^3 + ax + b = 0$, & $x^3 + ax = 0$, ſunt diviſores.

Exemplum III.

In genere ſit reſolutio $x = \sqrt[n]{a + \sqrt{-b^2}} + \sqrt[n]{a - \sqrt{-b^2}}$, exterminentur irrationales quantitates & reſultat æquatio ($n \times n$) dimenſionum, cujus æquatio

$x^n - n \times \overline{a^2 - b^2}^{\frac{1}{n}} x^{n-2} + n \times \frac{n-3}{2} \overline{a^2 - b}^{\frac{2}{n}} x^{n-4} - n \times \frac{n-4}{2} \times \frac{n-5}{3} \overline{a^2 - b}^{\frac{3}{n}} x^{n-6} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n+7}{4} \overline{a^2 - b}^{\frac{4}{n}} x^{n-8} -$ &c. $= 0$, eſt diviſor.

Exemplum IV.

Sit reſolutio $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4} \ldots\ldots$ $A\sqrt[n]{p^{n-4}} + B\sqrt[n]{p^{n-3}} + C\sqrt[n]{p^{n-2}} + D\sqrt[n]{p^{n-1}}$. exterminentur irrationales quantitates & reſultat æquatio

$$x^n - n \times \overline{aD + bC + cB + dA}, \text{ \&c. } x^{n-2} + \text{ \&c.} = 0.$$

Hinc deduci poteſt cubicæ æquationis reſolutio.

Sit reſolutio

$x = a\sqrt[3]{p} + \sqrt[3]{p^2}$; reducatur, ita ut exterminentur irrationales quantitates, & reſultat æquatio $x^3 - 3apx - p^2 - a^3p = 0$, ſit cubica æquatio

tio $x^3+Ax-B=o$, cujus reſolutio requiritur; aſſumatur $A=-3ap$ & $B=p^2+a^3p$ & conſequenter $a=\frac{-A}{3p}$, $p^2+a^3p=p^2-\frac{A^3}{27p^3}p=B$ vel $p^4-Bp^2=\frac{A^3}{27}$ & $p^2=\pm\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}+\frac{1}{2}B$, exinde datur p & ſolutio eadem fit ac Cardani.

Aliquantulum diverſa eſt reductio ſi modo reſolutio ſit $x=\sqrt[3]{p}+a\sqrt[3]{p^2}$; exterminentur irrationales quantitates & reſultat $x^3-3apx-p-a^3p^2=o$; fiat hæc eadem ac cubica æquatio $x^3+Ax-B=o$ & fit $p^2-Bp-\frac{A^3}{27}=o$, æquatio quadratica

COROLLARIUM I.

Ex hoc problemate inveniri poſſunt quilibet æquationum numerus quarum dantur reſolutiones.

Aſſumatur reſolutio, & transformetur hæc æquatio ita ut deleantur irrationales quantitates & reſultat æquatio, cujus datur reſolutio. Et ſic in quamplurimis caſibus reſolvi poteſt data æquatio.

EXEMPLUM I.

Sit data æquatio

$$x^3+Ax+B=o.$$

Aſſumatur $x=\sqrt[3]{\alpha}+\sqrt[3]{\beta}$ & conſequenter

$$x^3=\alpha+\beta+3\sqrt[3]{\alpha\beta}\times\overline{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$$

$$+Ax= \qquad A\times\overline{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$$

$$B= \quad B.$$

Supponatur $\alpha+\beta+B=o$, & $3\sqrt[3]{\alpha\beta}+A=o$, unde $\alpha+\beta=-B$ & $\alpha\beta=\frac{-A^3}{27}$. & α & β erunt duæ radices quadraticæ æquationis

$$z^2+Bz-\frac{A^3}{27}=o.$$

EXEMPLUM II.

Sit data æquatio

$$x^4+qx^2+rx+s=o.$$

aſſumatur pro eius reſolutione

$$x=\sqrt[4]{\alpha}+\sqrt[4]{\beta}+\sqrt[4]{\gamma}.$$

&

&

$$x^4 = \alpha+\beta+\gamma+6\overline{\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}+\alpha^{\frac{1}{2}}\gamma^{\frac{1}{2}}+\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}+4\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}\times\overline{\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}+\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}+\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}}+$$
$$8\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}\times\overline{\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}+\gamma^{\frac{1}{4}}}$$

$$qx^2 = q\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}+2q\times\overline{\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}+\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}+\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}}$$

$$rx = r\times\overline{\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}+\gamma^{\frac{1}{4}}}$$

$$s = s$$

Supponantur $4\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}+2q=0$,

$\alpha+\beta+\gamma+6\times\overline{\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}+\alpha^{\frac{1}{2}}\gamma^{\frac{1}{2}}+\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}+q\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}+s=0$

& $8\,\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}+r=0$. e primâ æquatione ſequitur

$\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}=-\frac{q}{2}$, quo valore in ſecundâ ſubſtituto, reſultat

$\alpha+\beta+\gamma+6\times\frac{\frac{q^2}{4}-\alpha+\beta+\gamma}{2}-\frac{q^2}{2}+s=0$, unde ſequitur

$\alpha+\beta+\gamma=\frac{s}{2}+\frac{q^2}{8}$. Per lemma ſecundum capitis primi inveniri poteſt

$\alpha\beta+\alpha\gamma+\beta\gamma=+\frac{qr^2}{64}+\frac{s^2}{16}+\frac{q^2s}{32}+\frac{q^4}{256}$ Sequitur & e tertiâ æquatione

$\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}=\frac{-r}{8}$; quâ reductâ, fit $\alpha\beta\gamma=\frac{r^4}{4096}$: & conſequenter æquatio, cujus radices (z) ſunt α, β, γ erit

$$z^3-\overline{\frac{s}{2}+\frac{q^2}{8}}\,z^2+\overline{\frac{qr^2}{64}+\frac{s^2}{16}+\frac{q^2s}{32}+\frac{q^4}{256}}\,z-\frac{r^4}{4096}=0.$$

Datæ æquationis quatuor radices erunt reſpective

$$\sqrt[4]{\alpha}+\sqrt[4]{\beta}+\sqrt[4]{\gamma},$$
$$-\sqrt[4]{\alpha}-\sqrt{-1}\sqrt[4]{\beta}+\sqrt{-1}\sqrt[4]{\gamma},$$
$$\sqrt{-1}\sqrt[4]{\alpha}-\sqrt[4]{\beta}-\sqrt{-1}\sqrt[4]{\gamma},$$
$$-\sqrt{-1}\sqrt[4]{\alpha}+\sqrt{-1}\sqrt[4]{\beta}-\sqrt[4]{\gamma}.$$

EX-

EXEMPLUM III.

Sit data æquatio

$$x^5+qx^3+rx^2+sx+t=0$$

aſſumatur pro ejus reſolutione $x=\sqrt[5]{\alpha}+\sqrt[5]{\beta}+\sqrt[5]{\gamma}+\sqrt[5]{\delta}$; ex calculo inveni æquationem biquadraticam, cujus radices ſunt α, β, γ, δ, irrationales quantitates neceſſario habere & conſequenter per hanc methodum haud reſolvi poteſt data æquatio.

EXEMPLUM IV.

Sit reſolutio

$$x=\left(a+\left(b+\left(c+\left(d+e^{\frac{1}{m}}\right)^{\frac{1}{n}}\right)^{\frac{1}{o}}\right)^{\frac{1}{p}}\right)^{\frac{1}{q}}+f$$

ita transformetur hæc æquatio, ut exterminentur irrationales quantitates & reſultat æquatio

$$\left(\left(\left(\left((x-f)^{q}-a\right)^{p}-b\right)^{o}-c\right)^{n}-d\right)^{m}-e=0$$

Et ſic de infinitis hujuſmodi exemplis.

Ab exemplo primo & ſecundo patet cubicarum æquationum generalem reſolutionem eſſe reſolutionem ſex vel novem dimenſionum æquationum.

Ab exemplo ſecundo corollarii patet eam biquadraticæ æquationis reſolutionem eſſe reſolutionem (64) dimenſionum æquationis.

Datâ igitur in genere (n) dimenſionum æquatione

$$x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+\&c.=0$$

ejus reſolutio erit reſolutio æquationis, cujus dimenſiones ſunt multò altiores.

Subſequens, ut mihi videtur, eſt unica methodus generalem æquationum reſolutionem inveniendi.

Aſſu-

Aſſumatur reſolutio, plures quam datæ æquationis (n) dimenſiones incognitas quantitates habens, transformetur hæc æquatio ita ut exterminentur irrationales quantitates, & reſultat æquatio, cujus data æquatio eſt diviſor; unde erui poſſunt (n) æquationes, plures quam (n) incognitas quantitates habentes, quibus reductis, reſultat æquatio duas vel plures incognitas quantitates habens.

Incognitarum reſultantis æquationis quantitatum inveniantur rationales radices, ſibi ipſis correſpondentes; quibus pro ſuis valoribus ſubſtitutis, reſultant incognitæ quantitates quæſitæ.

Facile patebit reſolutio, quam exigat quæcunque data æquatio. Sed in multarum dimenſionum æquationibus calculus requiritur fere infinite laborioſus.

COROLLARIUM II.

Datâ æquatione, invenire, annon reſolvi poteſt extractionis radicis quadraticæ, cubicæ &c, ope.

Datâ æquatione, dantur etiam ejus dimenſiones, aſſumatur generaliter reſolutio, quæ ſolummodo habet radicem quadraticam, cubicam &c, & ita ut, ſi modo exterminentur ſurdæ quantitates, reſultat æquatio datarum dimenſionum.

Reſultans & data æquatio æquales inter ſe fiant, & fit problema.

EXEMPLUM.

Invenire, annon data quatuor vel octo dimenſionum æquatio reſolvi poteſt extractionis radicis quadraticæ ope.

Aſſumatur generaliter reſolutio, quæ radices habet quadraticas, ita ut reſultans æquatio quatuor vel octo &c. habeat dimenſiones, & erit reſolutio

$$x = \sqrt{a + m\sqrt{b}} + n\sqrt{b} + c \text{ vel } x = \sqrt{a + m\sqrt{b + n\sqrt{c}} + o\sqrt{c}} + r\sqrt{b + n\sqrt{c}} + s\sqrt{c} + d$$

reducantur hæ æquationes, ita ut exterminentur irrationales quantitates & reſultat æquatio

$$x^4 - 4cx^3 - \overline{2n^2b - 6c^2 + 2a}\,x^2 - \overline{4c^3 - 4ac}$$

$+$

$$\overline{+4mnb-4n^2cb}\, x+\left.\overline{\begin{matrix}n^2b+c^2\\-a\end{matrix}}\right)^2-\overline{m-2nc}^{\,2}\times b=0.$$

vel x^8-8cx^7 &c. $=0$.

Fiant reſultans & data æquatio inter ſe æquales & fit.

COROLLARIUM.

Hinc reduci poteſt æquatio $2n$, $4n$, $8n$, &c. dimenſionum in æquationem (n) dimenſionum extractionis quadraticæ radicis ope:

EXEMPLUM I.

Sit data æquatio

$$x^{2n}-px^{2n-1}+qx^{2n-2}-rx^{2n-3}+\text{\&c.}=0.$$

aſſumatur

$$x^n-Px^{n-1}+Qx^{n-2}\text{\&c.}=\sqrt{a}\times\overline{\overset{1}{p}x^{n-1}-\overset{1}{q}x^{n-2}}$$

transformetur hæc æquatio, ita ut exterminetur irrationalis quantitas $(\sqrt{a})$ & fit

$$x^{2n}-2Px^{2n-1}+\overline{2Q+P^2-a\overset{1}{p}{}^2}\times x^{2n-2}\text{ \&c.}=0.$$

Fiat hæc æquatio eadem ac data, & conſequenter earum coefficientes inter ſe æquales

i. e. $p=2P, q=2Q+P^2-a\overset{1}{p}{}^2$ & ſic deinceps. Ex æquationibus reſultantibus inveniri poſſunt rationales incognitarum coefficientium ſi quos forte habent, valores.

EXEMPLUM II.

Sit data æquatio

$$x^{4n}-px^{4n-1}+qx^{4n-2}-rx^{4n-3}+\text{\&c.}=0.$$

aſſumatur

$$x^n-Px^{n-1}+\overline{Qx^{n-2}+\sqrt{a}\times\overline{\overset{1}{p}x^{n-1}+\overset{1}{q}x^{n-2}}}\text{ \&c.}=$$

$$\sqrt{\overline{b+m\sqrt{a}}}\times\overset{11}{p}x^{n-1}+\overset{11}{q}x^{n-2}+\text{\&c.}$$

transformetur hæc æquatio, ita ut exterminentur irrationales quantitates, & reſultantis æquationis coefficientes correſpondentibus datæ æquationis coefficientibus fiant æquales; & reſultantium æquationum incognitarum quantitatum quicunque rationales valores erunt coefficientes quæſitæ.

CAP. V.

PROB. IX.

DUAS æquationes in unam transformare, ut incognita quantitas exterminetur.

1. Collocentur ſingulæ æquationis termini juxta incognitæ quantitatis exterminandæ dimenſiones, ita ſubducatur alterutra æquatio aut ejus multiplex de alterâ, & reliquum aut ejus multiplex de ablatâ æquatione & ſic deinceps; ut tandem exterminetur incognita quantitas.

EXEMPLUM.

Sint duæ æquationes

$$y^2+\overline{3+2x}\times y+4x^2+5x+6=0$$
$$y^2+\overline{5+2x}\times y+10x^2+7x+12=0.$$

et operatio erit hujuſmodi

$$y^2+\overline{3+2x}\times y+4x^2+5x+6\ (y^2+\overline{5+2x}\times y+10x^2+7x+12)\ 1$$
$$\underline{y^2+\overline{3+2x}\times y+4x^2+5x+6}$$
$$2y\quad+6x^2+2x+6=0$$

$$2y+6x^2+2x+6\ (y^2+\overline{3+2x}\times y+4x^2+5x+6)\ \tfrac{1}{2}y+\frac{x-3x^2}{2}$$
$$\underline{y^2+\overline{3x^2+x+3}\times y}$$
$$\overline{x-3x^2}\times y+4x^2+5x+6$$
$$\underline{\overline{x-3x^2}\times y+\frac{x-3x^2}{2}\times\overline{6x^2+2x+6}}$$
$$4x^2+5x+6-\overline{x-3x^2}\times\overline{3x^2+x+3}=0$$

& transformantur duæ datæ æquationes in unam ($4x^2+5x+6-\overline{x-3x^2}\times\overline{3x^2+x+3}=0$) ut exterminetur incognita quantitas (y).

Et ſic de (m) æquationibus in unam transformandis, ut ($m-1$) incognitæ quantitates exterminentur.

COROLLARIUM.

Æquationes ſic derivatæ plerumque plures habent dimenſiones, quam neceſſario exigat problema.

2. Eſt

2. Eſt alia methodus transformandi duas æquationes in unam, ut incognita quantitas exterminetur

Sint duæ æquationes

$$y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + \&c. = 0$$
$$V - Ty + Sy^2 - Ry^3 + \&c. = 0$$

Sint etiam $\alpha, \beta, \gamma, \delta$, &c. prioris æquationis radices $(y,)$ quarum numerus eſt $(n;)$ quibus in poſteriori pro ſuo valore (y) ſubſtitutis, reſultant quantitates

$$V - T\alpha + S\alpha^2 - R\alpha^3 + \&c.$$
$$V - T\beta + S\beta^2 - R\beta^3 + \&c.$$
$$V - T\gamma + S\gamma^2 - R\gamma^3 + \&c.$$
$$V - T\delta + S\delta^2 - R\delta^3 + \&c.$$

multiplicentur hæ quantitates reſultantes continuo in ſeſe & contentum erit

$$\begin{array}{llll} V^n - V^{n-1}T \times \alpha & + V^{n-1}S \times \alpha^2 & & \&c. \\ \quad\quad\quad\quad \beta & \quad\quad\quad\quad \beta^2 & & \&c. \\ \quad\quad\quad\quad \gamma & \quad\quad\quad\quad \gamma^2 & & \&c. \\ \quad\quad\quad\quad \delta & \quad\quad\quad\quad \delta^2 & & \\ & \quad\quad\quad\quad \&c. & & \\ & + V^{n-2}T^2 \times \alpha\beta & & \&c. \\ & \quad\quad\quad\quad \alpha\gamma & & \\ & \quad\quad\quad\quad \alpha\delta & & \\ & \quad\quad\quad\quad \beta\gamma & & \&c. \\ & \quad\quad\quad\quad \&c. & & \end{array}$$

Ope lemmatum in primo capite inveniri poteſt hoc contentum, quod nihilo fiat æquale; & transformantur duæ æquationes in unam, ut exterminetur incognita quantitas (y).

EXEMPLUM.

Sint duæ æquationes

$$y^2 + \overline{3 + 2x} \times y + 4x^2 + 5x + 6 = 0.$$
$$y^2 + \overline{5 + 2x} \times y + 10x^2 + 7x + 12 = 0.$$

Sint α & β prioris æquationis radices (y); quibus pro ſuo valore (y) in poſteriori ſubſtitutis, reſultant quantitates

$$\alpha^2 + \overline{5 + 2x} \times \alpha + 10x^2 + 7x + 12$$
$$\beta^2 + \overline{5 + 2x} \times \beta + 10x^2 + 7x + 12$$

 quibus

quibus in ſeſe ductis, ſit rectangulum

$$\alpha^2\beta^2+\overline{5+2x}\times\alpha\beta\times\overline{\alpha+\beta}+\overline{10x^2+7x+12}\times\overline{\alpha^2+\beta^2}+\overline{5+2x}^2\times\alpha\beta+\overline{10x^2+7x+12}\times\overline{5+2x}\times\overline{\alpha+\beta}$$
$$+\overline{10x^2+7x+12}^2.$$

Ope lemmatis primi invenitur hoc rectangulum

$$\overline{4x^2+5x+6}^2-\overline{5+2x}\times\overline{4x^2+5x+6}\times\overline{3+2x}+\overline{10x^2+7x+12}\times\overline{3+2x}^2 \quad -2\times\overline{4x^2+5x+6}$$
$$+\overline{5+2x}^2\times\overline{4x^2+5x+6}-\overline{10x^2\times 7x+12}\times\overline{5+2x}\times\overline{3+2x}+\overline{10x^2+7x\times 12}^2$$

Fiat hoc rectangulum nihilo æquale & erit quæſita æquatio.

COR. I.

Hac methodo transformentur duæ æquationes, quarum dimenſiones ſunt n & m reſpective, duas incognitas quantitates (x & y) habentes, in unam, ut exterminetur una incognita quantitas (y,) & plures quam ($n\times m$) dimenſiones non habere poteſt reſultans æquatio.

COR. II.

Sit data æquatio

$$My^n+\overline{a+bx}\times y^{n-1}+\overline{c+dx+ex^2}\times y^{n-2}+\&c.=0$$
$$Ny^m+\overline{A+Bx}\times y^{m-1}+\overline{C+Dx+Ex^2}\times y^{m-2}+\&c.=0$$

quarum dimenſiones ſunt n & m, duas incognitas quantitates (x & y) habentes, rejiciantur harum æquationum omnes termini, in quibus incognitarum quantitatum (x & y) dimenſio non invenitur maxima & quantitates reſultantes erunt

$$My^n+bxy^{n-1}+ex^2y^{n-2}+\&c$$
$$Ny^m+Bxy^{m-1}+Ex^2y^{m-2}+\&c$$

ſi communem habent diviſorem hæ quantitates reſultantes; ($n\times m$) dimenſiones non habere poteſt reſultans æquatio, cujus radix eſt incognita quantitas (x vel y.)

COR. III.

Si æquatio reſultans, cujus radix eſt incognita quantitas (x,) ($n\times m$) dimenſiones habeat, ſumma radicum ſolummodo pendet e terminis priorum æquationum, in quibus incognitarum quantitatum (x & y) dimenſiones ſunt n & $n-1$, m & $m-1$; & ſic de ſummâ contentorum ſub ſingulis binis, ternis &c. radicibus.

COR. IV.

Si duæ vel tres vel plures incognitæ quantitatis (x) radices, ſint inter ſe æquales; æquatio cujus radix eſt (x,) deprimi poteſt in alteram pauciores dimenſiones habentem.

E X-

EXEMPLUM.

Sint duæ æquationes duas incognitas quantitates habentes

$$y^3+\overline{6+2x}\times y+12x^3+14x^2-7x+6=0$$
$$y^3+\overline{6+2x}\times y+9x^3+14x^2+14x-12=0$$

auferatur poſterior æquatio de priori, & reſiduum erit $3x^3-21x+18=0$. quæ eſt æquatio, cujus radix eſt incognita quantitas $(x.)$

Sed æquatio, per methodum ſupra traditam derivata, erit $\overline{3x^3-21x+18}^3=0$. in quâ tres radices incognitæ quantitatis (x) ter ſunt æquales; tres enim diverſi incognitæ quantitatis (y) valores, æquationis $(3x^3-21x+18=0)$ radici ſingulæ reſpondent.

Æquationis $(3x^3-21x+18=0)$ radices ſunt 1, 2 & -3; quibus pro ſuo valore (x) in datis æquationibus ſubſtitutis, reſultant tres æquationes

$$y^3+8y+25=0$$
$$y^3+10y+144=0$$
$$y^3 \quad -171=0.$$

unde ſequitur tres radices æquationis $(y^3+8y+25=0,)$ reſpondere radici (1) incognitæ quantitatis $(x,)$; & conſequenter tres radices incognitæ quantitatis (x) eſſe (1); & ſic de reliquis.

COR. V.

Transformentur duæ æquationes, duas incognitas quantitates $(x$ & $y)$ habentes & ſimiliter involventes, in unam; ut exterminetur altera incognita quantitas $(y,)$ & æquatio reſultans, cujus radix eſt (x) duplas habet dimenſiones æquationis, cujus radix eſt $(x+y)$ aut (x^2+y^2) &c.

DEMONSTRATIO.

Sint radices incognitæ quantitatis (x) reſpective

$\alpha, \beta, \gamma, \delta \ldots \pi, \rho, \sigma, \tau$; correſpondentes vero radices incognitæ quantitatis (y) reſpective

$\tau, \sigma, \rho, \pi \ldots \delta, \gamma, \beta, \alpha$. & incognitæ quantitatis $(z=x+y)$ reſpectivi valores ſunt

$\alpha+\tau, \beta+\sigma, \gamma+\rho, \delta+\pi \ldots \delta+\pi, \gamma+\rho, \beta+\sigma, \alpha+\tau$, unde omnis valor incognitæ quantitatis (z) alterum habet ſibi ipſi æqualem

lem ($\alpha+\tau$, $\alpha+\tau$, $\beta+\sigma$, $\beta+\sigma$, &c.) & conſequenter incognita quantitas (x aut y) duplum habet valorum numerum, quem incognita quantitas ($z=x+y$,) & exinde æquatio, cujus radix eſt (x aut y) duplas habet dimenſiones æquationis, cujus radix eſt ($z=x+y$.)

Hujuſce corollarii exempla præbent Moivræi duo problemata.

1. Sint duæ æquationes $\overline{x^2+y^2} \times \overline{x^4+y^4} = b$

$$\overline{x^2+y^2} \times \overline{x+y} = a$$

utriuſque æquationis terminorum, in quibus incognitarum quantitatum (x aut y) dimenſiones ſunt maximæ, communis eſt diviſor (x^2+y^2); unde ſequitur, æquationem, cujus radix eſt (x vel y) ($6\times 3=18$) dimenſiones non habere: ejus dimenſiones ſunt (12.); ſed hæ duæ æquationes incognitas quantitates (x&y) ſimiliter involvunt, & conſequenter æquatio, cujus radix eſt ($z=x+y$) ſolummodo ſex habet dimenſiones, & ex calculo ſequitur formulam ejus eſſe quadraticam.

2. Sint æquationes

$$x^4+x^3y+x^2y^2+xy^3+y^4=a$$

$$\overline{x^4+x^3y+x^2y^2+xy^3+y^4} \times \overline{x^4-x^3y+x^2y^2-xy^3+y^4}=b$$

harum æquationum termini, in quibus incognitarum quantitatum (x aut y) dimenſiones ſunt maximæ, communem habent diviſorem ($x^4+x^3y+x^2y^2+xy^3+y^4$), & conſequenter æquatio, cujus radix eſt (x aut y) non habet ($4\times 8=32$) dimenſiones; habet vero ($4\times 4=16$) dimenſiones; & eadem eſt ejus formula, ac octo dimenſionum æquatio, unde ejus negativæ & affirmativæ radices ſunt inter ſe æquales, ſed hæ duæ æquationes incognitas quantitates (x &y) ſimiliter involvunt, & conſequenter æquatio, cujus radix eſt ($z=x^2+y^2$) ſemper quatuor habet radices ſibi ipſis æquales, duas, quoniam ſimiliter involvunt incognitas quantitates (x & y) datæ æquationes, & duas reliquas, quoniam affirmativæ & negativæ radices (x) ſunt inter ſe æquales, unde æquatio, cujus radix eſt ($z=x^2+y^2$) non habere poteſt plures quam $\left(\frac{16}{4}=4\right)$ dimenſiones; & ex calculo ſequitur ejus formulam eſſe quadraticam.

3. Eſt etiam alia methodus transformandi duas æquationes in unam, ut incognita quantitas exterminetur.

Per

Per methodum ſerierum convergentium, inveniantur ſingulæ radices incognitæ quantitatis exterminandæ in unâ datâ æquatione, quibus pro ſuo valore in altera ſubſtitutis, & multiplicentur quantitates reſultantes continuo in ſeſe, contentum nihilo æquale fiat, & erit quæſita æquatio.

EXEMPLUM.

Sint duæ æquationes

$$y^2 - \overline{4x+2} \times y + 3x^2 + 4x + 4 = 0$$
$$y^2 - 3xy + 4x^2 + 6 = 0.$$

eas in unam transformare, ut exterminetur incognita quantitas (y).

Incognitæ quantitatis (y) duæ radices in priori æquatione per convergentes ſeries inveniuntur

$$y = x + 1 + \frac{3}{2x} + \frac{9}{8x^3} + \&c.$$

$$y = 3x + 1 - \frac{3}{2x} - \frac{9}{8x^3} - \&c.$$

quibus radicibus pro (y) in poſteriori æquatione ſubſtitutis, reſultant quantitates

$$2x^2 - x + \frac{11}{2} + \frac{3}{x} + \frac{9}{8x^2} \quad \&c.$$

$$4x^2 + 3x + \frac{5}{2} - \frac{3}{x} - \frac{9}{8x^2} \quad \&c.$$

ducantur hæ duæ quantitates reſultantes in ſeſe & rectangulum ſub iis contentum, erit

$8x^4 + 2x^3 + 24x^2 + 20x + 28$ quod nihilo fiat æquale, & erit æquatio quæſita

$$8x^4 + 2x^3 + 24x^2 + 20x + 28 = 0.$$

PROB. X.

Invenire coefficientium conſtitutionem, in datâ æquatione, unam vel plures incognitas quantitates habente.

1. Datâ æquatione ($x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}$ &c. $= 0$) unam incognitam quantitatem habente, invenire ejus coefficientium (p, q, r, s &c.) conſtitutionem.

Sint

Sint $\alpha, \beta, \gamma, \delta$ &c radices datæ æquationis, quarum numerus ſit (n,) quibus pro ſuo valore (x) ſubſtitutis, reſultant æquationes

$$\alpha^n - p\alpha^{n-1} + q\alpha^{n-2} - r\alpha^{n-3} + s\alpha^{n-4} \text{ \&c.} = 0$$
$$\beta^n - p\beta^{n-1} + q\beta^{n-2} - r\beta^{n-3} + s\beta^{n-4} \text{ \&c.} = 0.$$
$$\gamma^n - p\gamma^{n-1} + q\gamma^{n-2} - r\gamma^{n-3} + s\gamma^{n-4} \text{ \&c.} = 0.$$
$$\delta^n - p\delta^{n-1} + q\delta^{n-2} - r\delta^{n-3} + s\delta^{n-4} \text{ \&c.} = 0.$$

reducantur hæ æquationes in unam, ita ut exterminentur omnes præter unam (p) coefficientes, & ſequitur $p = \alpha + \beta + \gamma + \delta +$ &c, eodem modo invenitur $q = \alpha\beta + \alpha\gamma + \beta\gamma +$ &c, $r = \alpha\beta\gamma +$ &c & ſic deinceps.

2. Datâ æquatione

$$y^n + \overline{a + bx} \times y^{n-1} + \overline{c + dx + ex^2} \times y^{n-2} + \overline{f + gx + hx^2 + kx^3} \times y^{n-3} \text{ \&c.} = 0.$$

duas incognitas quantitates (x & y) habente, invenire coefficientium conſtitutionem.

Sint incognitæ quantitatis (y) radices $\alpha, \beta, \gamma, \delta$ &c, & incognitæ quantitatis (x) correſpondentes radices π, ρ, σ, τ &c. & ſingulæ incognitæ quantitatis (x & y) ſit numerus correſpondentium radicum $n \times \frac{n+3}{2}$; ſubſtituantur hæ correſpondentes radices pro ſuis reſpectivis valoribus (x & y,) & reſultant æquationes

$$\alpha^n + \overline{a + b\pi} \times \alpha^{n-1} + \overline{c + d\pi + e\pi^2} \times \alpha^{n-2} + \overline{f + g\pi + h\pi^2 + k\pi^3} \times \alpha^{n-3} \text{ \&c.} = 0$$
$$\beta^n + \overline{a + b\rho} \times \beta^{n-1} + \overline{c + d\rho + e\rho^2} \times \beta^{n-2} + \overline{f + g\rho + h\rho^2 + k\rho^3} \times \beta^{n-3} \text{ \&c.} = 0$$
$$\gamma^n + \overline{a + b\sigma} \times \gamma^{n-1} + \overline{c + d\sigma + e\sigma^2} \times \gamma^{n-2} + \overline{f + g\sigma + h\sigma^2 + k\sigma^3} \times \gamma^{n-3} \text{ \&c.} = 0$$
$$\delta^n + \overline{a + b\tau} \times \delta^{n-1} + \overline{c + d\tau + e\tau^2} \times \delta^{n-2} + \overline{f + g\tau + h\tau^2 + k\tau^3} \times \delta^{n-3} \text{ \&c.} = 0.$$

harum æquationum numerus per hypotheſin æqualis eſt fractioni $\frac{n \times \overline{n+3}}{2}$; et numerus incognitarum coefficientium (a, b, c, d, e &c,) eidem quantitati æqualis eſt; reducantur hæ $\left(\frac{n \times \overline{n+3}}{2}\right)$ æquationes in unam, ita ut exterminentur omnes præter unam (a) incognitæ coefficientes, & conſtat ex quot & qualibus membris *i.e.* ex quâ radicum functione iſta una (a) coefficiens conſtat; & ſic de reliquis.

COROLLARIUM. II.

1. Sit numerus (m) correſpondentium radicum incognitarum quantitatum (x & y) æquationis, cujus dimenſiones ſunt (n,) major quam

$n \times$

$n \times \overline{\frac{n+3}{2}}$; tum ex quibuſque $\left(n \times \overline{\frac{n+3}{2}}\right)$ correſpondentibus radicibus incognitarum quantitatum (x & y) oritur æquationis conſtitutio, *i, e.* coefficientes quæſitæ æquationis.

2. Sit numerus (m) correſpondentium radicum minor quam $\left(n \times \overline{\frac{n+3}{2}},\right)$ & infinitis modis dari poſſunt diverſæ conſtitutiones æquationis, vel pro $\left(n \times \overline{\frac{n+3}{2}} - m\right)$ coefficientibus ſcribi poſſunt quilibet numeri.

COR. III.

Sint duæ æquationes, duas incognitas quantitates (x & y) habentes, quarum dimenſiones ſunt n & m reſpective, et ſit (n) minor quam m; tum æquationis n dimenſionum ita aſſumi poſſunt coefficientes, ut quicunque $\left(n \times \overline{\frac{n+3}{2}}\right)$ numeri, ſint radices incognitæ quantitatis (x), & ſimiliter quicunque $\left(n \times \overline{\frac{n+3}{2}}\right)$ numeri, ſint correſpondentes radices incognitæ quantitatis (y); æquationis vero m dimenſionum coefficientes infinitis modis variari poſſint:

Sint n & m inter ſe æquales, tum haud aſſumi poſſunt coefficientes utriuſque æquationis, ita ut quicunque $\left(n \times \overline{\frac{n+3}{2}}\right)$ numeri ſint radices incognitæ quantitatis (x), & quicunque etiam $\left(n \times \overline{\frac{n+3}{2}}\right)$ numeri ſint correſpondentes radices incognitæ quantitatis (y): ſi enim hoc fit, utraque æquatio eadem fiet: ſed aſſumi poſſunt coefficientes infinitis modis, ita ut quicunque $\left(n \times \overline{\frac{n+3}{2}} - 1\right)$ numeri ſint reſpective correſpondentes valores incognitarum quantitatum (x & y.)

EXEMPLUM.

Sint duæ quadraticæ æquationes, duas incognitas quantitates habentes

$$y^2 + \overline{a+bx} \times y + c + dx + ex^2 = 0 \ \&$$
$$y^2 + \overline{A+Bx} \times y + C + Dx + Ex^2 = 0$$

* H in

in utraque æquatione ſunt quinque $\left(2 \times \frac{\overline{2+3}}{2}\right)$ incognitæ coefficientes, ſubſtituantur pro incognitis quantitatibus (x & y) quicunque quinque numeri reſpective in utrâque æquatione, & oriuntur ex utrâque eædem quinque ſimplices æquationes, unde patet, quod utraque æquatio eadem erit; ergo tantum ſubſtituti poſſunt in utrâque æquatione quatuor correſpondentes valores incognitarum quantitatum (x & y), quibus ſubſtitutis, reſultant ex utrâque quadraticâ, quatuor ſimplices æquationes quinque incognitas coefficientes habentes, & conſequenter in utrâque pro unâ incognitâ coefficiente aſſumi poteſt quicunque numerus, at cavendum eſt, ne idem numerus pro eâdem coefficiente in utriſque æquationibus aſſumatur.

Sint quatuor valores incognitæ quantitatis (y) 0, 2, 4, 2, & quatuor correſpondentes valores incognitæ quantitatis (x) 0, 1, 1, 2, quibus ſubſtitutis, & in utrâque quadraticâ pro unâ coefficiente (b & B) aſſumatur reſpective $+4$ & -2: & reſultant ex primâ quadraticâ, quatuor ſimplices æquationes

$$c=0$$
$$4+2a+8+e+d=0$$
$$16+4a+16+e+d=0$$
$$4+2a+16+4e+2d=0$$ unde ſequitur

primam æquationem eſſe $y^2+\overline{4x-10}\times y-8x^2+16x=0$; & eodem modo invenitur ſecunda $y^2-\overline{2x+4}\times y-2x^2+10x=0$.

COROLLARIUM.

Numerus correſpondentium radicum incognitarum quantitatum (x & y), quæ generaliter aſſumi poſſunt, æqualis eſt numero incognitarum datæ æquationis coefficientium.

Eadem eſt methodus ratiocinandi de pluribus æquationibus plures incognitas quantitates habentibus: ſed obſervandum eſt æquationem n dimenſionum, m incognitas quantitates involventem,

$\overline{n+1} \times \frac{\overline{n+2}}{2} \times \frac{\overline{n+3}}{3} \dots \frac{\overline{n+m}}{m} - 1$ incognitas coefficientes habere poſſe.

LEM-

LEMMA VI.

Sint duæ æquationes, duas incognitas quantitates (x & y) *habentes, & ſint* α, β, γ, δ *&c. radices incognitæ quantitatis* (y), *&* π, ρ, σ, τ *&c. correſpondentes radices incognitæ quantitatis* (x)

invenire ſummam

$$\alpha^m \times \pi^n + \beta^m \times \rho^n + \gamma^m \times \sigma^n + \delta^m \times \tau^n + \text{\&c.}$$

Supponantur $y^m \times x^n = z$, & in datis æquationibus pro (y) ſubſtituatur ejus valor $\left(\frac{z^{\frac{1}{m}}}{x^{\frac{n}{m}}}\right)$; & transformentur duæ æquationes reſultantes in unam, ut exterminetur incognita quantitas (x); & reſultat æquatio cujus radix eſt z, & cujus radicum ſumma erit ſumma quæſita.

LEMMA VII.

Invenire ſummam omnis hujuſmodi producti

$$\alpha^m \pi^a \beta^n \varrho^b \gamma^o \sigma^c \text{ \&c.} + \alpha^m \pi^a \beta^m \rho^b \delta^o \tau^c + \text{\&c.}$$

ſint

$$p^m \times Q^a = \alpha^m \pi^a + \beta^m \rho^a + \gamma^m \sigma^a + \delta^m \tau^a + \text{\&c.}$$
$$p^n \times Q^b = \alpha^n \pi^b + \beta^n \rho^b + \gamma^n \sigma^b + \delta^n \tau^b + \text{\&c.}$$
$$p^o \; Q^c = \alpha^o \pi^c + \beta^o \rho^c + \gamma^o \sigma^c + \delta^o \tau^c + \text{\&c.}$$

&c. &c. &c.

$$p^{m+n} \times Q^{a+b} = \alpha^{m+n} \pi^{a+b} + \beta^{m+n} \varrho^{a+b} + \gamma^{m+n} \sigma^{a+b} + \text{\&c.}$$
$$p^{m+o} \times Q^{a+c} = \alpha^{m+o} \pi^{a+c} + \beta^{m+o} \rho^{a+c} + \gamma^{m+o} \sigma^{a+c} + \text{\&c.}$$
$$p^{n+o} \times \gamma^{b+c} = \alpha^{n+o} \pi^{b+c} + \beta^{n+o} \rho^{b+c} + \gamma^{n+o} \sigma^{b+c} + \text{\&c.}$$

&c. &c. &c.

$$p^{m+n+o} \times Q^{a+b+c} = \alpha^{m+n+o} \pi^{a+b+c} + \beta^{m+n+o} \rho^{a+b+c} + \text{\&c.}$$

&c. &c, &c.

& ſic obſervatâ lemmatis ſecundi ſubſtitutionis lege; ſcribatur

$$A = p^m \; Q^a \times p^n Q^b \times p^o Q^c \text{ \&c.}$$

$$B = \frac{p^{m+n} Q^{a+b} \times A}{p^m Q^a \times p^n Q^b} + \frac{p^{m+o} Q^{a+c} \times A}{p^m Q^a \times p^o Q^c} + \text{\&c.}$$

$$C = \frac{p^{m+n+o} Q^{a+b+c} \times A}{p^m Q^a \times p^n Q^b \times p^o Q^c} + \frac{p^{m+n+r} Q^{a+b+d} \times A}{p^m Q^a \times p^n Q^b \times p^r Q^d} + \text{\&c.}$$

$D=$

$$D = \frac{p^{m+n+o+r} \mathcal{Q}^{a+b+c+d} \times A}{p^m \mathcal{Q}^a \times p^n \mathcal{Q}^b \times p^o \mathcal{Q}^c \times p^r \mathcal{Q}^d} + \&c.$$

$$BB = \frac{p^{m+n} \mathcal{Q}^{a+b} \times p^{o+r} \times \mathcal{Q}^{c+d} \times A}{p^m \mathcal{Q}^a \times p^n \times \mathcal{Q}^b \times p^o \mathcal{Q}^c \times p^r \mathcal{Q}^d} + \frac{p^{m+o} \mathcal{Q}^{a+c} \times p^{n+r} \times \mathcal{Q}^{b+d} \times A}{p^m \mathcal{Q}^a \times p^n \mathcal{Q}^b \times p^o \mathcal{Q}^c \times p^r \mathcal{Q}^d} + \&c.$$

&c. &c.

ſumma quæſita eſt

$$A - B + 2C - 1 \times 2 \times 3 D + 1 \times 2 \times 3 \times 4 E - \&c.$$
$$+ \ 1 \times 1 BB - 1 \times 1 \times 2 \ BC + \&c.$$

lex, quam obſervat hæc ſeries, e lemmate ſecundo petenda eſt.

PROB. XI.

Transformare duas æquationes, duas incognitas quantitates (x & y) *habentes in unam, cujus incognita quantitas aſſignabilem habet relationem ad incognitas datarum æquationum quantitates,* (x & y).

Aſſumantur radices incognitarum quantitatum tanquam cognitæ, *i. e.* ſint radices incognitæ quantitates (x); $\alpha, \beta, \gamma, \delta$, &c. & correſpondentes radices incognitæ quantitatis (y); π, ρ, σ, τ &c. tum ex aſſignabili relatione inveniantur radices quæſitæ quantitatis; & conſequenter per lemma præcedens, ſumma ejus radicum, quæ eſt coefficiens ſecundi æquationis termini.

Ducantur quæque duæ, tres &c. radices quæſitæ quantitatis in ſeſe & lemmatis præcedentis ope, reſultantium contentorum inveniri poſſunt aggregata, quæ ſunt reſpective coefficientes, tertii, quarti &c. quæſitæ æquationis terminorum: unde ſequuntur quæſitæ æquationis coefficientes & conſequenter æquatio ipſa.

COROLLARIUM.

Eodem fere modo transformari poſſunt duæ vel plures æquationes in duas vel plures alias, quarum incognitæ quantitates aſſignabiles habent relationes ad incognitas datarum æquationum quantitates.

PROB. XII.

Invenire datarum æquationum radices, quæ datam inter ſe habent relationem.

1. Sint radices incognitarum quantitatum (x & y) ſigillatim datæ, vel algebraicam relationem inter ſe habeant correſpondentes incognitarum

nitarum quantitatum (x & y) radices, pro incognitis quantitatibus (x & y) ſubſtituantur earum valores & quantitatum reſultantium communes diviſores erunt radices quæſitæ.

2. Si æquationes datam relationem exprimentes diverſas ejuſdem incognitæ quantitatis radices involvunt, a ſecundo problemate & capite tertio ſequitur ejus ſolutio.

COROLLARIUM.

Datis duabus æquationibus, duas vel plures incognitas quantitates habentibus, ex hoc problemate inveniri poteſt, utrum duæ vel plures radices ſuntinter ſe æquales; facilius vero per ſubſequentem methodum.

Sint duæ æquationes duas incognitas quantitates habentes

$$y^n + \overline{a+bx} \times y^{n-1} + \overline{c+dx+ex^2} \times y^{n-2} \text{ \&c.} = 0.$$

$$y^m + \overline{A+Bx} \times y^{m-1} + \overline{C+Dx+Ex^2} \times y^{m-2} + \text{\&c.} = 0.$$

ſint duæ radices incognitæ quantitatis (y) inter ſe æquales & erit

$$ny^{n-1} + n-1 \times \overline{a+bx} \times y^{n-2} + \text{\&c.} = 0$$

$$\text{\&} \quad my^{m-1} + \overline{m-1} \times \overline{A+Bx} \times y^{m-2} + \text{\&c.} = 0.$$

Inveniantur harum æquationum communes diviſores, & inveniuntur duæ vel plures æquales radices.

PROB. XIII.

Transformare (m) *æquationes, vel* (m+1) *vel plures incognitas quantitates habentes in unam, ita ut exterminentur* (m—1) *incognitæ quantitates, etiamque ſi æquationis reſultantis incognitæ quantitates vel ſint rationales quantitates vel integri numeri, exterminatæ etiam erunt vel rationales quantitates vel integri numeri.*

1. Per noni problematis methodum primam transformentur (m) æquationes in ($m-1$), ita ut exterminetur incognita quantitas (x), et ſi in operatione reſultat æquatio ſimplicem ſolummodo poteſtatem incognitæ quantitatis (x) habens, tum ita reducantur (m) æquationes in ($m-1$), ut exterminetur incognita quantitas (x), etiamque ſi ($m-1$) æquationum reſultantium incognitæ quantitates ſint rationales quantitates, exterminata (x) erit rationalis quantitas; et ſic iteratâ operatione exterminentur tandem ($m-1$) incognitæ quantitates.

2. Si prior methodus haud ita transformat æquationes, ut exigat problema, recurrendum eſt ad ſubſtitutionem *i. e.* pro incognitâ quantitate

titate (x) ſubſtituatur rationalis functio reliquarum incognitarum quantitatum & novæ quantitatis (v); & deinde inveniendum eſt, annon per methodum præcedentem transformationem deſideratam admittit.

3. In genere, pro ſingulâ incognitâ quantitate exterminandâ, ſubſtituatur rationalis functio æquationis reſultantis incognitarum quantitatum, ſcribantur hæ rationales functiones pro ſuis valoribus in datis æquationibus, æquationum reſultantium ſinguli termini nihilo vel inter ſe æquales fiant, & ex iis colligi poſſunt quæſitæ coefficientes.

PROB. XIV.

Datis (m) *æquationibus,* (m+1) *incognitas quantitates habentibus, invenire rationales vel integros earum valores, ſibi ipſis correſpondentes.*

Sit (z) quæcunque aſſumpta quantitas, pro ſingulâ ($m-1$) incognitâ quantitate ſubſtituatur in genere rationalis functio aſſumptæ quantitatis (z,) & æquationum reſultantium fiant ſinguli termini nihilo reſpective æquales; reſultantium æquationum ope inveniri poſſunt rationalium functionum aſſumptæ quantitatis (z) coefficientes quæſitæ.

COROLLARIUM.

Methodo in Capite tertio datâ, inveniri poſſunt innumeræ (m) æquationes, ($m+1$) incognitas quantitates habentes, ita ut correſpondentes radices incognitarum quantitatum ſint rationales functiones aſſumptæ quantitatis (z).

Si hæc methodus haud invenit incognitarum quantitatum rationales valores, ſibi ipſis correſpondentes, tum haud admittunt generalem regulam æquationes, ſed in multis caſibus approximatio innumeros invenit rationales valores.

Sint etiam (m) æquationes, eundum (m) numerum incognitarum quantitatum habentes, & earum rationales valores per approximationem inveniri poſſunt.

EXEMPLUM.

Sit irrationalis quantitas

$$a+\sqrt{b}$$

invenire ejus radicem (n) poteſtatis, quæ ſit hujuſce formulæ ($x+\sqrt{y}$.)

Inveniatur quam proxime valores quantitatis

$\overline{a+\sqrt{b}}^{\frac{1}{n}} = x+\sqrt{y}$ & $\overline{a^2-b}^{\frac{1}{n}} = x^2-y$ & conſequenter

a^2-

$$\frac{\overline{a^2-b}^{\frac{1}{n}}}{\overline{a+\sqrt{b}}^{\frac{1}{n}}} = \frac{x^2-y}{x+\sqrt{y}} = x-\sqrt{y} \text{ unde ſequitur}$$

$$\frac{\overline{a+\sqrt{b}}^{\frac{1}{n}} + \overline{a-\sqrt{b}}^{\frac{1}{n}}}{2} = x \ \& \ \frac{\overline{a+\sqrt{b}}^{\frac{1}{n}} - \overline{a-\sqrt{b}}^{\frac{1}{n}}}{2} = \sqrt{y},$$

quibus valoribus (x & y) quam proxime inventis, pro iis vel earum poteſtatibus aſſumantur proximi integri numeri & fit exemplum.

Et ſic de extrahendis radicibus diverſis poteſtatibus affectis, ſed ſubſequentis problematis ope omnia hujuſmodi problemata reſolvi poſſunt.

Si vero ſit incognita quantitas, vel incognitæ quantitates, dati numeri, aut datorum numerorum diviſores; approximationis ope earum integri valores minime inveniri poſſunt, nulla enim regula pro inveniendis numerorum diviſoribus dari poteſt.

Et ſic de multis aliis caſibus & ad integros & rationales pertinentibus.

Æquationum, quarum incognitæ vel incognitarum quantitatum integri valores, ſunt datorum numerorum diviſores, facile eſt dare exempla.

Ex. gr. Sint (n) æquationes ($n+m$) incognitas quantitates habentes, & ſint omnes poteſtates (m) incognitarum quantitatum in datis æquationibus, multiplicatæ in incognitam quantitatem (x,) ita transformentur (n) æquationes in unam, ut exterminentur omnes præter (m) incognitas quantitates prædictas, & (x) in quam ductæ fuerunt, & invenitur numerus, cujus diviſor eſt incognita quantitas (x.) &c.

PROB. XV.

Datis duabus æquationibus, duas incognitas quantitates (x & y) *habentibus, invenire rationales incognitæ, quantitatis* (y), *ſi quas forte habet, radices.*

1. In datis æquationibus pro incognitâ quantitate (y) ſubſtitue tres vel plures terminos hujus progreſſionis arithmeticæ $-1, 0, 1, 2,$ &c; duas quantitates reſultantes ex ſingulâ ſubſtitutione colloca juxta dimenſiones literæ (x,) ita ut illi termini primum locum occupent, in quibus litera iſta (x) eſt plurimarum dimenſionum &c. & perpetuâ ablatione minoris a majori & reliqui de ablato, exterminetur incognita quan-

quantitas (x.) & ſit arithmetica ſeries $z+1, z, z-1, z-2$ diviſores numerorum reſultantium reſpective, tum diviſor (z) erit rationalis incognitæ quantitatis (y) quæſita radix.

EXEMPLUM.

Sint duæ æquationes

$$y^2+\overline{1+3x}\times y+2x^2+4x-18=0.$$
$$y^2+\quad 3x\times y+3x^2+4x-17=0.$$

pro y ſubſtituatur (0) & reſultant quantitates,

$$2x^2+4x-18$$
$$3x^2+4x-17$$

transformentur hi termini, ita ut exterminetur (x), & operatio eſt.

$$\overline{2x^2+4x-18}\times 3 = 6x^2+12x-54$$
$$\overline{3x^2+4x-17}\times 2 = \underline{6x^2+8x-34}$$ auferatur poſterior de priori & reſiduum erit $4x-20$ tum

$$\overline{2x^2+4x-18}\times 2 = 4x^2+8x-36$$
$$\overline{4x-20}\times x = \underline{4x^2-20x}$$
reſiduum $28x-36$
$$\overline{4x-20}\times 7 = \underline{28x-140}$$
reſiduum $= 104.$

Et ſic ſubſtituantur 1 & -1 & numeri reſultantes erunt 512, 972 reſpective, quos cum omnibus eorum diviſoribus colloco

1	512	1, 2, 4, 8, 16, 32, 64, 128, 256, 512,	1
0	104	1, 2, 4, 8, 13, 26, 52, 104	2
—1	972	1, 2, 3, 4, &c.	3

unica tantum eſt arithmetica progreſſio $z-1, z, z+1$, (1, 2, 3), cujus communis differentia eſt (1,) ergo numerus qui ſtat e regione termini (0,) eſt rationalis quantitatis (y) radix.

COR. I.

Eodem ferè modo quadratici, cubici &c. diviſores duarum vel plurium æquationum duas vel plures incognitas quantitates habentium inveniri poſſunt.

COR. II.

Sint duæ æquationes, duas incognitas quantitates habentes, & ſit (y) integer numerus, vel fractio; (x) etiam erit integer numerus vel fractio, niſi duæ, tres vel plures incognitæ quantitatis (x) radices ſint inter ſe æquales.

2. Eſt

2. Eſt etiam alia methodus hoc problema ſolvendi:

Sint duæ æquationes duas incognitas quantitates (x & y) habentes

$$T - Sy + Ry^2 - Qy^3 + \ldots y^n = 0$$

$$t - sy + ry^2 - qy^3 + \ldots y^m = 0$$

in quibus T, S, R, Q &c. t, s, r, &c. denotant reſpective rationales integras functiones literæ (x), & ſit z rationalis integra radix quantitatis (y) quæſita, tum

$$\frac{T}{z},\ \frac{t}{z},\ \frac{Sz-T}{z^2},\ \frac{sz-t}{z^2},\ \frac{Rz^2-Sz+T}{z^3},\ \frac{rz^2-sz+t}{z^3}\ \text{\&c.}$$

integri erunt numeri.

LIBER SECUNDUS.

CAP. I.

PROB. I.

Fig. 1. *D**ATA æquatione relationem inter abſciſſam* (AP = x) *& ejus ordinatas* (PM = y) *definiente, invenire æquationem, relationem definientem, inter abſciſſam* (ap), *quæ ſecat abſciſſam* (AP) *ad datum angulum* (p*NP*), *& in dato puncto* (N), *& ordinatas* (pM &c.) *eam* (ap) *ſecantes in dato angulo* (MpN).

Sit radius (1) & ſinus angulorum *APM*, *pNP* & *MpN* reſpective s, S & σ, tum ſinus angulorum *NrP* & *rMp* erunt,

$$s\sqrt{1-S^2}+S\sqrt{1-s^2}=c, \;\&\; \sigma\sqrt{1-\overline{s\sqrt{1-S^2}+S\sqrt{1-s^2}}^{\,2}} + \overline{s\sqrt{1-S^2}+S\sqrt{1-s^2}}\times\sqrt{1-\sigma^2}=C:$$

ſubſtituantur pro lineis *NA*, *Na*, *ap*, *pM* reſpective a, b, z, v, & erit

$$z+b-\frac{C}{c}v=Nr, \text{ \& conſequenter } \frac{cz+cb-Cv}{s}+a=x$$

$$\& \; y=rM+rP=\frac{\sigma v}{c}+\frac{Sz+Sb}{s}-\frac{CS}{cs}v=\frac{\sigma s-CS}{cs}\times v+\frac{Sz}{s}+\frac{Sb}{s}$$

quibus quantitatibus pro ſuis valoribus (x & y) in datâ æquatione ſubſtitutis, reſultat æquatio relationem inter abſciſſam (*ap*) & ejus ordinatas (*pM*) definiens.

COR. I.

Fig. 2. Datâ æquatione relationem inter abſciſſam (*AP*, x) & ejus ordinatas (*PM*, v) definiente, invenire æquationem definientem relationem inter abſciſſam ($Ap=z$) & ejus ordinatas ($pM=v$).

Sint

Sint ſinus angulorum *APM* & *ApM* reſpective s, S; & ſinus anguli (pMP) erit $s\sqrt{1-S^2} + S\sqrt{1-s^2}$; pro ordinata *PM*, y, in datâ æquatione ſubſtituatur ejus valor $\frac{S \times v}{s}$, & pro abſciſſâ *AP*, (x) ſubſtituatur $z \pm \frac{s\sqrt{1-S^2} + S\sqrt{1-s^2}}{s} v$, & fit.

COR. II.

Datis ut in ſuperioribus, invenire æquationem, relationem inter abſciſſam *Np*, z, & ordinatam *Mp*, v, exprimentem: Fig. 3.

Sint ſinus datorum angulorum *APM* & *PNp* reſpective s & S; tum ſinus anguli (MpN) erit $s\sqrt{1-S^2} + S\sqrt{1-s^2}$; pro abſciſſâ *AP* (x) in datâ æquatione ſcribatur ejus valor $\frac{s\sqrt{1-S^2} + S\sqrt{1-s^2}}{s} z$ $+ AN$ (a) & pro ordinatâ (y); $v + \frac{Sz}{s}$ & reſultat æquatio quæſita.

COR. III.

Sit æquatio relationem definiens inter abſciſſam (x) & ejus ordinatas (y) hujuſce formulæ

$$y^n + ay^{n-1} + \overline{c + dx} \times y^{n-2} \&c. = 0.$$

vel $$y^n + bxy^{n-1} + \overline{c + ex^2} \times y^{n-2} \&c. = 0.$$

vel etiam ſi modo ſit æquatio

$$y^n + \overline{a + bx}y^{n-1} + \overline{c + dx + ex^2}y^{n-2} + \overline{f + gx + hx^2 + kx^3}y^{n-3} + \&c. = 0.$$

& duæ æquationes $bq^{n-1} + 2eq^{n-2} + 3kq^{n-3} + \&c. = 0.$

& $aq^{n-2} + dq^{n-2} + hq^{n-3} + \&c. = 0.$

communem habeant diviſorem, tum abſciſſa erit diameter, ſin aliter vero non.

Hoc corollarium conſtat ſubſtitutione in corollario primo datâ.

COR. IV.

Sit æquatio relationem inter abſciſſam (NP) & ejus ordinatas (PM) exprimens, (n) dimenſionum Fig. 4.

$$Ay^n + \overline{a + bx}\, y^{n-1} + \overline{c + dx + ex^2}\, y^{n-2} + \&c. = 0.$$

Per hoc problema transformetur hæc æquatio in æquationem relationem inter abſciſſam (Np, z) & ejus ordinatas $(Mp = v)$ exprimentem:

Sint anguli $(MpN$ & $NPM)$ & conſequenter eorum ſinus s & σ, dati.

Æquationis, ex hâc transformatione reſultantis, termini (zv^{n-1}) coefficiens erit

$$n\times\overline{\frac{\sigma s - CS}{cs}}\Big|^{n-1}\times\frac{S}{s}A+\overline{\overline{\frac{\sigma s-CS}{cs}}\Big|^{n-1}\times\frac{c}{s}-\overline{n-1}\times\overline{\frac{\sigma s-CS}{cs}}\Big|^{n-2}\times\frac{C}{s}}\times\frac{S}{s}\times b+\overline{\overline{\frac{\sigma s-CS}{cs}}\Big|^{n-2}\times\frac{-2Cc}{s^2}+\overline{n-2}\times\overline{\frac{\sigma s-CS}{cs}}\Big|^{n-3}\times\frac{\overline{C^2}}{s^2}}\times e \text{ \&c.}$$

Pro C & c ſubſtituantur in hac coefficiente earum valores, & fiat coefficiens nihilo æqualis, in reſultanti æquatione, a ſurdis liberatâ, incognita quantitas (S), ſurget ad ($8n$) dimenſiones, & tot poſſunt eſſe diverſi valores incognitæ quantitatis S, (*i.e.* ſinus anguli PNp,) quot ſunt ejus dimenſiones; quatuor vero valores incognitæ quantitatis in eundem angulum referuntur, & conſequenter ($2n$) poſſunt eſſe diametri, quæ ſecant ſuas ordinatas ad datum angulum (NpM).

Hinc ſequitur curvam (n) dimenſionum non poſſe plures quam (n) axes (*i.e.* diametros ſuas ordinatas ad rectum angulum ſecantes) habere.

COROLLARIUM V.

Æquationis exprimentis relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y), ſint termini, in quibus quantitatum (x & y) dimenſiones inveniuntur maximæ, ($\overline{Ay+Bx}^{n}$); & omnes curvæ diametri erunt inter ſe parallelæ, & vice verſâ omnis linea cuicunque diametro parallela, erit diameter.

Pro $y+\frac{B}{A}x$ ſubſtituatur (v), & pro (x) Sz, & reſultat æquatio, in quâ terminus, in quo dimenſiones abſciſſæ & ordinatæ inveniuntur maximæ, erit v^n, *i. e.* æquatio reſultans eſt hujuſce formulæ

$$v^n+\alpha v^{n-1}+\overline{\gamma+\delta z}\,v^{n-2}+\text{\&c.}=0.$$

& ſimiliter æquatio, relationem inter quamcunque abſciſſam huic parallelam & ejus correſpondentes ordinatas exprimens, erit ejuſdem formulæ.

Sin æquatio relationem inter abſciſſam & correſpondentes ordinatas exprimens, ſit hujus formulæ, tum abſciſſa erit diameter.

Linea,

Linea, huic diametro haud parallela, non erit diameter: æquatio enim relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens, non erit prædictæ formulæ, & conſequenter abſciſſa non erit diameter.

In hoc caſu duæ & haud plures poſſunt eſſe diametri, quæ datum angulum cum ſuis ordinatis faciunt, ni rectus eſt angulus, & tum unica ſolummodo erit diameter.

Non conſtare poteſt curva, cujus omnes diametri ſunt parallelæ, ex hyperbolicis cruribus, ni omnes eorum aſymptotes ſint inter ſe parallelæ.

Non conſtat ex parabolicis cruribus, ni omnia crura ſunt concava vel convexa eâdem directione.

Conſtant hæc e ſubſequentibus.

COROLLARIUM VI.

Æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, ſint nulli termini, in quibus inveniuntur $(n-1)$ dimenſiones, & omnes curvæ diametri ſeſe interſecant in puncto, quod eſt abſciſſæ vertex. Fig. 4.

Per corollarium ſecundum transformetur hæc æquatio in alteram relationem inter abſciſſam Ap (z), utcunque inclinatam ad datam abſciſſam (AP), & ejus correſpondentes ordinatas pM (v) exprimentem; & ſemper reſultat æquatio hujus formulæ

$$v^n + bzv^{n-1} + \overline{c+ez^2}\, v^{n-2} \text{ \&c.} = o$$

& per corollarium tertium abſciſſam hujus æquationis eſſe diametrum conſtat, & conſequenter omnes lineæ per hoc centrum ductæ, erunt diametri.

Sit curva (n) dimenſionum, (n) habens aſymptotes, ſeſe ſecantes in uno eodemque puncto; tum iſtud punctum erit curvæ centrum.

Si centrum non eſt vertex abſciſſæ, ſubſtituatur pro $(AP=x)$ abſciſſâ, $(ap+\alpha=z+\alpha)$, & pro ordinatâ (y) $(pM+\beta=v+\beta)$, & ita aſſumantur quantitates α & β, ut nihilo fiant æquales omnes termini, in quibus dimenſiones ſunt $(n-1)$, & vertex novæ abſciſſæ erit centrum. Fig. 5.

Sit

Si curva habet parabolicum crus hujuſce formulæ ($y^{n+r}=\alpha x^{m}$), in quâ (y & x) ſunt abſciſſæ & ordinatæ reſpective, & (r) non eſt minor quam (2), tum neceſſario habet generale centrum.

Curva (n) dimenſionum non habere poteſt plures quam ($n-1$) parallelas diametros vel diametros ſeſe interſecantes in uno eodemque puncto, ni omnes ejus diametri ſunt parallelæ vel centrum habent generale.

Fig. 2. Sit enim angulus APM rectus, & pro ordinata (PM) ſubſtituatur Sv, & pro ejus abſciſſâ $z-\sqrt{1-S^2}v$ & reſultat coefficiens termini (zv^{n-1})

$$bS^{n-1}-2eS^{n-2}\sqrt{1-S^2}+3kS^{n-3}\times\overline{1-S^2}\ \&c.$$

fiat hæc coefficiens nihilo æqualis, & ita transformetur reſultans æquatio, ut exterminentur irrationales quantitates; & reſultat æquatio ($2\times\overline{n-1}$) dimenſionum, duos vero valores habens, inter ſe æquales, alterum affirmativum, alterum vero negativum, & conſequenter ad eandem ordinatarum inclinationem referuntur hi duo valores, & numerus angulorum & parallelarum diametrorum non major eſt quam ($n-1$,) ni omnes ſunt parallelæ.

Eodem modo ratiocinari licet de diametris in uno eodemque puncto ſeſe ſecantibus.

COROLLARIUM VII.

Curva poteſt habere unicam ſolummodo diametrum, quam dico generalem diametrum, cui etiam lineæ utcunque inclinatæ erunt ordinatæ.

Hujuſmodi vero diametrum habet curva, ſi æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, termini, in quibus dimenſiones inveniuntur maximæ, ſint $\left(\overline{Ay+Bx}^{\,n}\right)$ & nulli ($n-1$) dimenſionum inveniuntur termini.

Hoc in caſu, e prædictis conſtat, curvam & habere omnes ſuas diametros inter ſe parallelas, & omnes ſeſe ſecare in uno eodemque puncto, & conſequenter curva unicam ſolummodo habet diametrum.

Transformetur enim æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens, in alteram, in quâ, terminus, in quo inveniuntur maximæ dimenſiones, ſit ſolummodo (v^n), & ($n-1$) dimenſionum ſint nulli termini tum abſciſſa erit generalis diameter.

Si curva habet parabolicum crus hujus generis ($y^{m+r}=ax^m$), in quâ r major eſt quam (3), tum generalem habet diametrum.

Hæ

Hæ ſunt diametri primi generis, quarum ordinatæ curvam ſecant in (n) punctis, & eorum numerus, uno quidem caſu excepto, eſt infinitus.

Si vero ordinatæ ſecant curvam in ($n-1$, $n-2$, $n-3$, &c.) punctis reſpective, & hæ ordinatæ ita ſecantur per rectas, ut ſumma ſegmentorum in omnibus utrinque ſint inter ſe æqualis, tum hæ lineæ reſpective dici poſſunt diametri ſecundi, tertii, &c. generis; eodem modo diſtingui poſſunt abſciſſæ diverſorum generum.

COROLLARIUM VIII.

Non poteſt ſumma diametrorum, ſecundi, tertii, quarti &c. generum in curva (n) dimenſionum, major eſſe quam $\left(\frac{n}{3}\right)$.

Sit enim æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas cum eâ rectum angulum facientes

$$y^n+\overline{a+bx}\,y^{n-1}+\overline{c+dx+ex^2}\,y^{n-2}+\overline{f+gx+hx^2+kx^3}\,y^{n-3}+\text{\&c.}=0.$$

Transformetur hæc æquatio, ita ut abſciſſæ eædem maneant, ordinatarum vero mutetur inclinatio: *i.e.* per corollarium primum pro ordinata (y), ſubſtituatur Sv; & pro abſciſſa (x), $z+\sqrt{1-S^2}\,v$, & reſultat æquatio, Fig. 2.

$$S^n+b\sqrt{1-S^2}\times S^{n-1}+e\,\overline{1-S^2}\times S^{n-2}+k\,\overline{1-S^2}^{1\frac{1}{2}}\times S^{n-3}+\text{\&c.}\,v^n$$

Ut fiat diameter vel abſciſſa ſecundi, generis, neceſſe eſt, ut evaneſcat terminus (v^n), & conſequenter ejus coefficiens nihilo fiat æqualis, *i. e.*

$$S^n+bS^{n-1}\times\sqrt{1-S^2}+eS^{n-2}\times\overline{1-S^2}+\text{\&c.}=0.$$

quâ æquatione reductâ, aſcendit æquatio reſultans ad $2n$ dimenſiones, cujus radix $2n$ habet valores, duo vero ſinus in eandem inclinationem referuntur, & conſequenter (n) ſolummodo poſſunt eſſe inclinationes, in quibus ordinatæ curvam (n) dimenſionum haud ſecant in (n) punctis vel poſſibilibus vel impoſſibilibus.

Ut fiat diameter vel abſciſſa tertii generis, neceſſe eſt, ut evaneſcat prædictus terminus (v^n), & termini (zv^{n-1} & v^{n-1}); coefficientes vero horum terminorum reſpective ſunt

$$bS^{n-1}+2eS^{n-2}\sqrt{1-S^2}+3kS^{n-3}\times\overline{1-S^2}+\text{\&c.}$$

$$\text{\&}\quad aS^{n-1}+dS^{n-2}\sqrt{1-S}+hS^{n-3}\times\overline{1-S^2}+\text{\&c.}$$

Fiant hæ coefficientes nihilo reſpective æquales, & reſultant duæ æqua-

æquationes $2 \times \overline{n-1}$ dimenſionum, & conſequenter numerus valorum incognitæ quantitatis (S) eſt $2\,\overline{n-1}$; & exinde numerus diverſarum inclinationum ad abſciſſam (AP) erit $n-1$; ſed ſecundæ æquationis radices ſunt limites inter primæ æquationis radices, & conſequenter non plures poſſunt eſſe diverſæ radices harum duarum æquationum inter ſe æquales, quam $\frac{n}{2}$.

Terminorum ($z^2 v^{n-2}$, $z v^{n-2}$, v^{n-2}) coefficientes ſunt reſpective

$$eS^{n-2} + 3kS^{n-3}\sqrt{1-S^2} + \&c.$$
$$dS^{n-2} + 2bS^{n-3}\sqrt{1-S^2} + \&c.$$
$$cS^{n-2} + gS^{n-3}\sqrt{1-S^2} + \&c.$$

Fiant hæ coefficientes & prima

$$S^n + bS^{n-1}\sqrt{1-S^2} + eS^{n-2}\,\overline{1-S^2} + k\,S^{n-3}\,\overline{1-S^2}^{\,1\frac{1}{2}} + \&c.$$

nihilo reſpective æquales, & ex hâc & primâ æquatione ſequitur haud plures diverſas radices eſſe harum duarum æquationum quam $\frac{n}{3}$, & ſic deinceps.

Non poſſunt igitur eſſe plures quam $\frac{n}{m}$ inclinationes parallelarum ordinatarum, quæ ſecant curvam in ($n-m$) punctis.

Non poſſunt eſſe plures quam $\frac{n}{m+1}$ diametri, cujus ordinatæ ſecant curvam in ($n-m$) punctis.

Non poſſunt eſſe plures quam (n) diverſæ ordinatarum inclinationes curvam in (n) punctis haud ſecantium vel poſſibilibus vel impoſſibilibus.

Non poteſt igitur ſumma diametrorum ſecundi, tertii, &c. ordinum eſſe major quam $\frac{n}{3}$.

Diametri vel potius axes nonnunquam definiuntur eſſe lineæ, quæ ita ſecant ordinatas ſibi ipſis ad rectum angulum inſiſtentes, ut ordinatæ utrinque ſint inter ſe æquales.

CO-

COROLLARIUM IX.

Sit curva (*MNS*), cujus linea (*AP*) eſt hujuſmodi diameter, tum ordinatæ utrinque ſunt inter ſe æquales *i.e.* negativi & affirmativi ordinatæ valores ſunt æquales, & æquatio erit hujus formulæ Fig. 6.

$$y^{2n}+\overline{c+dx+ex^2}\,y^{2n-2}+\overline{f+gx+hx^3+kx^3+lx^4}\times y^{2n-4}+\&c.=0$$

Ita ſcribatur hæc æquatio, ut termini progrediantur ſecundum dimenſiones incognitarum quantitatum (x & y) iis contentarum,

$$\alpha+\beta x+\gamma y^2+\delta x^2+\epsilon xy^2+\zeta x^3+ny^4+\theta x^2y^2+ix^4+\&c.=0$$

transformetur hæc æquatio, ut relationem inter abſciſſam ($Ap=z$) & ejus ordinatas ($pM=v$) rectum angulum cum abſciſſa (Ap) facientes, exprimat, *i.e.* ſi modo (q) ſit ſinus anguli PAp, pro

PM (y) ſubſtituatur $qz+\sqrt{1-q^2}\times v$ &

pro AP (x) ſcribatur $\sqrt{1-q^2}\times z-qv$, quibus quantitatibus pro ſuis valoribus ſubſtitutis, & reſultantis æquationis terminis, in quibus inveniuntur impares ordinatæ (v) dimenſiones, nihilo æquales eſſe ſuppoſitis, reſultant æquationes, e quibus cognoſci poſſunt diametri.

Ex. gr. Terminus unius dimenſionis eſt (βx), in eo pro (x) ſcribatur $\sqrt{1-q^2}\,z-qv$, & reſultat $\beta\sqrt{1-q^2}\,z-\beta qv$; ſed v eſt impar ordinatæ dimenſio, & conſequenter βqv nihilo debet eſſe æqualis, & exinde vel,

1. Sit β data quantitas & conſequenter $q=0$, tum curva non habet aliam diametrum præter datam, vel,

2. Sit $\beta=0$, & nulli ſint termini, in quibus una ſolummodo continetur dimenſio.

In terminis duarum dimenſionum ($\gamma y^2+\delta x^2$) pro (x & y) ſcribantur prædicti valores, & reſultant

$$\begin{array}{llll} \gamma\times\overline{1-q^2}\times v^2 & +2\gamma q\sqrt{1-q^2}\,zv & +\gamma q^2 & z^2 \\ \delta\times\quad q^2 & -2\delta q\sqrt{1-q^2} & +\delta\,\overline{1-q^2} & \end{array}$$

ut fiat (Ap) diameter vel axis, terminus $2\times\overline{\gamma-\delta}\times q\sqrt{1-q^2}\,zv$, in quo impar eſt dimenſio ordinatæ, debet eſſe nihilo æqualis.

Si vero $\overline{\gamma-\delta}\times q\sqrt{1-q^2}$ nihilo eſt æqualis; tum, ſi $\overline{\gamma-\delta}$ ſit data quantitas, vel q vel $\sqrt{1-q^2}$ nihilo eſt æquale, & q vel $=0$ vel $=1$. *i.e.* quoad hoc curva poteſt habere duas diametros ſecum angulum rectum facientes.

Si $\gamma - \delta = o$, tum quoad hoc curva poteſt habere infinitas hujuſmodi diametros; & hic caſus verus eſt in circulis, in quibus infinitæ dantur diametri.

In terminis $(\varepsilon x y^2 + \zeta x^3)$ pro $(x$ & $y)$ ſcribantur prædicti valores, & reſultant

$$-\varepsilon q \times \overline{1-q^2} \times v^3 + \varepsilon \overline{1-3q^2} \times \sqrt{\overline{1-q^2}}\, z v^2 + \varepsilon \times \overline{2q-3q^3}\, z^2 v + \varepsilon \overline{1-q^2}|^{\frac{1}{2}} z^3$$

$$-\zeta q^3 \qquad + \qquad 3\zeta q^2 \sqrt{\overline{1-q^2}} \qquad -3\zeta q \times \overline{1-q^2} + \zeta \times \overline{1-q^2}|^{\frac{1}{2}}$$

ſed ut fiat diameter, neceſſe eſt, ut evaneſcant termini, $(\overline{\varepsilon q \times \overline{1-q^2} + \zeta q^3}\, v^3$, $\overline{2\varepsilon q - 3\varepsilon q^3 - 3\zeta q \times \overline{1-q^2}}\, z^2 v)$, in quibus inveniuntur impares ordinatæ (v) dimenſiones; e priori vero termino ſequitur $\varepsilon = \frac{-\zeta q^2}{1-q^2}$, quâ quantitate pro ſuo valore in ſecundâ æquatione ſubſtitutâ, reſultat æquatio $\frac{-\zeta q^2}{1-q^2} \times \overline{2q-3q^3} - 3\zeta q \times \overline{1-q^2} = o$, quâ reductâ, ſequitur vel $q = o$ vel $q^2 = \frac{3}{4}$, & $q = \sqrt{\frac{3}{4}} =$ ſin. 60°; & conſequenter, ſi in datâ æquatione ſint quicunque termini, tres habentes dimenſiones ordinatæ & abſciſſæ, tum non poteſt habere curva plures quam tres diametros; & ſi tres ſint diametri, ſeſe interſecant in angulo 60°; & ε vel $= -3\zeta$ vel $= \zeta$; poſterior pars hoc modo deduci poteſt; ab æquatione $\varepsilon q \times \overline{1-q^2} + \zeta q^3 = o$ conſtat $\frac{\varepsilon}{\varepsilon - \zeta} = q^2$; quo valore in æquatione $\varepsilon \times \overline{2-3q^2} - 3\zeta \overline{1-q^2} = o$ ſubſtituto, reſultat æquatio $\varepsilon \times \overline{2 - \frac{3\varepsilon}{\varepsilon-\zeta}} - 3\zeta \times \overline{1 - \frac{\varepsilon}{\varepsilon-\zeta}} = o$, quâre ductâ, fit $\varepsilon^2 + 2\zeta\varepsilon - 3\zeta^2 = o$, cujus radices (ε) ſunt vel $= -3\zeta$ vel $= \zeta$.

In terminis $(ny^4 + \theta y^2 x^2 + i x^4)$ quatuor dimenſionum ſubſtituantur pro $(x$ & $y)$ eædem quantitates, quæ prius ſubſtitutæ fuerunt, & quantitatum reſultantium fiant illæ, in quibus inveniuntur impares ordinatæ dimenſiones nihilo æquales, & exinde ſequitur, ſi $2n = 2i = \theta$; tum quoad hos terminos infinitæ dari poſſunt diametri, ſin illæ quantitates haud ſunt inter ſe æquales, tum non plures quam

quatuor

quatuor poſſunt eſſe diametri; & ſi quatuor ſunt diametri, angulum inter ſe faciunt 45° graduum.

Duæ poſſunt eſſe etiam ſolummodo diametri; ſed hæc reſtrictio oritur e terminis duarum dimenſionum abſciſſæ & ordinatæ.

In genere, ſint curvæ (*n*) dimenſionum, quæ (*n*) habent diametros, & ſi modo ſint (*n*) diametri, anguli, quos hæ diametri inter ſe faciunt, erunt $\frac{180°}{n}$.

Si modo (*n*) eſt primus numerus, tum habet unam vel (*n*) vel nullam diametrum omnis hujuſmodi curva: & curva imparium dimenſinum non poteſt habere parem diametrorum numerum.

Conſtat methodus omnia hujus generis inveniendi.

Hæc facile demonſtrari poſſunt e geometricis principiis.

Hinc conſtat, ſi curva duas vel plures habet diametros, tum generale habet centrum.

Hinc etiam conſtat, ſi ducantur per centrum lineæ æquales angulos primo, tertio, quinto &c. axe facientes, tum hæ diametri æquales angulos cum ſuis ordinatis faciunt.

Primus, tertius, quintus &c. axes erunt inter ſe æquales.

Eodem modo, quo inveniuntur axes, deduci poſſunt diametri earum ordinatas ad datum angulum ſecantes, & ſic deinceps.

COROLLARIUM X.

Sunt curvæ (*n*) dimenſionum, quæ ſecare poſſunt rectam lineam in (*n*) punctis; ſunt etiam curvæ (2*n*) parium dimenſionum, quæ nullam interſecant rectam lineam in pluribus quam (2) punctis, & imparium dimenſionum, quæ nullam interſecant rectam lineam in pluribus quam (3) punctis; ſed obſervandum eſt, ut par erit maximus numerus punctorum, in quibus recta linea ſecare poteſt curvam parium dimenſionum, & impar, imparium dimenſionum.

COROLLARIUM XI.

Sit curva *MmM*, cujus æquatio relationem inter abſciſſam (*AP*) & ejus correſpondentes ordinatas (*PM*) exprimens eſt, Fig 7.

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = o.$$

ſint termini, in quibus maximæ inveniuntur dimenſiones incognitarum quantitatum (*y* & *x*)

$$Ay^n + bxy^{n-1} + ex^2y^{n-} + \&c.$$

 du-

ducantur duæ aliæ lineæ (PC, Pc) curvam in (n) punctis, & ſeſe interſecantes in puncto abſciſſæ (P), & cum eâ angulos facientes, quorum ſinus ſunt q & r reſpective, ſit (s) ſinus anguli (MPA), quem ordinatæ cum ſuâ diametro faciunt; & C, c ſinus angulorum, quos lineæ ſeſe interſecantes, in puncto (P) cum ordinatis (PM) faciunt, tum contentum ſub (n) lineis inter punctum (P) & curvam in directione (Pc) interceptis, erit ad contentum ſub (n) lineis inter punctum (P) & curvam in directione (PC) interceptis, in eadem ratione, quam habet

$$Aq^n + bq^{n-1}C + eq^{n-2}C^2 + \&c. \text{ ad } Ar^n + br^{n-1}c + er^{n-2}c^2 + \&c.$$

COROLLARIUM XII.

Datis duabus æquationibus eaſdem dimenſiones habentibus, relationem inter abſciſſas & earum correſpondentes ordinatas exprimentibus, invenire annon hæ duæ curvæ ſunt eædem.

Pro x & y in alterâ æquatione ſubſtituantur $nx + qv + r$ & $mz + ov + p$, & fiant reſultantis & datæ æquationis coefficientes eædem; ſi hoc poſſibile ſit, per problema inveniatur, utrùm ordinatarum & abſciſſæ mutatio admittit hujuſmodi ſubſtitutionem, & fit.

Hic innumeras hujuſmodi propoſitiones adjicere licet.

PROB. II.

Datâ æquatione relationem inter curvæ abſciſſam & ejus ordinatas exprimente, invenire ejus crura in infinitum abeuntia.

Per infinitas ſeries inveniantur ſinguli ordinatæ (y) valores; terminis vero ſecundum abſciſſæ (x) dimenſiones progredientibus, ita ut abſciſſæ (x) dimenſiones fiant continuo minores i. e. ſit

$$y = Ax^m + \&c. + P + \frac{Q}{x^r} + \&c.$$

crus, quod deſignat hæc æquatio, erit parabolicum vel hyperbolicum, prout (m) eſt unitas vel (o), necne.

COROLLARIUM I.

Sit æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens,

$$y^n + \overline{a + bx}\, y^{n-1} \overline{+ c + dx + ex^2}\, y^{n-2} + \&c. = o.$$

Sint

Sint termini in quibus maximæ ordinatæ & abſciſſæ (x & y) inveniuntur dimenſiones,

$$y^n + bxy^{n-1} + ex^2y^{n-2} + \&c.$$ fiant hi termini nihilo æquales, & reſultat

$$y^n + bxy^{n-1} + ex^2y^{n-2} + \&c. = 0.$$

Sint omnes radices hujus æquationis impoſſibiles, & curva nullum habet crus in infinitum progrediens;

Sint $(2m)$ radices impoſſibiles, & $(n-2m)$ poſſibiles & inæquales, ſint $\overline{(n-2m)}$ poſſibiles radices ordinatæ (y) αx, βx, γx, δx, &c. reſpective, *i. e.*

$y = \alpha x + \&c.$

$y = \beta x + \&c.$

&c. &c. & conſequenter curva habet $(n-2m)$ aſymptotes $(2n-4m)$ hyperbolica crura habentes.

Sint poſſibiles radices inter ſe æquales & eodem modo e ſubſequentibus terminis datæ æquationis inveniri poteſt formula ſeriei, a quâ deſignatur ordinata (y).

COROLLARIUM II.

Sit (n) æqualis $m + 2o + 3r + 4s + 5t + \&c.$ & facile deduci poteſt curva (n) dimenſionum, quaſcunque (m) lineas aſymptotes, quæcunque (o) parabolica conici generis crura, (r) ſecundi generis parabolica, & ſic deinceps habens.

Sit enim $(AP = x)$ abſciſſa, & $PM\,(y)$ ejus correſpondens ordinata, & ſint æquationes ad lineas aſymptotes

$$y + px + q = 0$$

$$y + \overset{\prime}{p}x + \overset{\prime}{q} = 0$$

$$y + \overset{\prime\prime}{p}x + \overset{\prime\prime}{q} = 0$$ & æquatio ad parabolam conici generis $\overline{y + \alpha x}^2 + \beta y + \gamma x + \delta = 0$ & ſic deinceps, e multiplicatione horum terminorum in ſeſe conſtant dimenſiones, ad quas aſſurgit data æquatio, & termini ipſi, in quibus inveniuntur (n) & $(n-1)$ dimenſiones.

PROB. III.

Datâ æquatione relationem inter abſciſſam & ejus correſpondentes ordinatas exprimente

$$Ay^n + \overline{a + bx}\,y^{n-1} + \overline{c + dx + ex^2}\,y^{n-2} + \&c. = 0.$$

inve-

invenire, qui ſit valor abſciſſæ, quando duo ordinatæ (y) *valores ſiunt inter ſe æquales.*

Si duo ordinatæ valores ſunt inter ſe æquales, tum

$$nAy^{n-1} + \overline{n-1} \times \overline{a+b}\,xy^{n-2} + \overline{n-2} \times \overline{c+dx+ex^2}\,y^{n-3} + \&c. = 0.$$

transformentur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (y), & reſultat æquatio, cujus radix eſt abſciſſa, quando duo ordinatæ valores fiunt inter ſe æquales.

Subſtituantur hæ radices pro ſuo valore (x) in duabus datis æquationibus, & æquationum reſultantium communes diviſores erunt valores ordinatarum, quando duo ſunt inter ſe æquales.

COROLLARIUM I.

Curvæ crura, cujus æquatio deſignans relationem inter ejus abſciſſam & ordinatas eſt

$$nAv^{n-1} + \overline{n-1} \times \overline{a+b}\,xv^{n-2} + \overline{n-2} \times \overline{c+dx+ex^2}\,v^{n-3} + \&c. = 0$$

ſemper interponentur inter crura curvæ, cujus æquatio relationem inter abſciſſam & ejus ordinatas exprimens, eſt

$$y^n + \overline{a+bx}\,y^{n-2} + \overline{c+dx+ex^2}\,y^{n-3} + \&c. = 0.$$

COROLLARIUM II.

Hinc inveniri poteſt, quando duo quatuor &c. ordinatæ valores fiunt impoſſibiles.

Inveniatur enim, quando duo valores fiunt æquales, & quando duo bis fiunt æquales, & ſic deinceps, & invenitur, quando duo, quatuor &c. ordinatæ valores fiunt impoſſibiles.

COROLLARIUM III.

Hinc etiam inveniri poſſunt puncta duplicia, triplicia, &c. & puncta conjugata, & ovales.

Inveniatur, quando duo, tres, quatuor & ordinatæ (y) valores fiunt inter ſe æquales.

Si duo, quatuor, ſex &c. ordinatæ valores fiunt inter ſe æquales, & utrinque ſint correſpondentes ordinatæ impoſſibiles, tum erit purum punctum conjugatum, ſimplex, duplex &c.

Sint præcedentes ordinatæ utrinque poſſibiles, & curva habet punctum duplex, quadruplex &c.

Si

Si ordinatæ partim ſunt poſſibiles & partim impoſſibiles utrinque, tum ſingula utrinque numerum æqualium radicum unitate auget, ſi vero ordinata tangit curvam, tum auget numerum æqualium ordinatarum per duo, & ſic de punctis contrariæ flexuræ, cuſpidibus.

Si plures ſunt ordinatæ æquales, quam requirit figura, tum datur punctum ad curvam.

Si duo vel plures ordinatæ valores bis fiunt æquales, & quarum abſciſſæ reſpective ſunt PA & $P\overset{\prime}{A}$, & inter eas ſint ordinatæ poſſibiles: ordinatæ vero PM & $\overset{\prime}{P}\overset{\prime}{M}$ eundem habeant numerum crurum utrinque puncta M & $\overset{\prime}{M}$ jacentium, tum jacet inter puncta M & $\overset{\prime}{M}$ ovalis. Fig. 8.

COROLLARIUM IV.

Sit punctorum numerus, in quibus duo, tria, quatuor &c. coeant reſpective (m, r, s, &c.) & N vel $= \sqrt{m+r+s+t+\frac{9}{4}} - \frac{3}{2}$, ſi ſit integer numerus, vel integer numerus ei proxime ſuperior, ſi modo ſit ſurda quantitas vel fractio, tum dimenſiones curvæ, in quâ includuntur hæc puncta, non poſſunt eſſe minores quam

$$\frac{N \times \frac{N+3}{2} + m + 2r + 3s + 4t + \&c.}{N}.$$

Et ſic de ovalibus.

PROB. IV.

Datâ æquatione

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = 0.$$

relationem inter abſciſſam (x) *& ejus ordinatas* (y), *exprimente, invenire æquationem relationem inter quaſcunque duas alias lineas quocunque modo ex iis derivatas exprimentem.*

Inveniantur æquationes relationem inter abſciſſam, ordinatas & lineas, inter quas requiritur relatio, exprimentes; transformentur hæ æquationes in unam, ita ut exterminentur omnes præter eas lineas, inter quas requiritur relatio, incognitæ quantitates, & fit problema.

EX-

EXEMPLUM.

Iiſdem poſitis, invenire æquationem relationem exprimentem inter lineas (*EM* & *FM*), a quibuſcunque duobus punctis (*E* & *F*) ad quodlibet curvæ punctum (*M*), ductas.

Sit ($AP=x$) abſciſſa, & ($PM=y$) ordinata, per puncta *E* & *F* ducantur lineæ *EK* & *FL* parallelæ ordinatis (*PM*), occurrentes abſciſſæ in punctis (*K* & *L*).

Scribantur pro *KA*, *EK*; *LA*, *FL*; *EM*, *FM* reſpective *E*, *F*, *G*, *H*, *v*, *z*, & pro ſinu anguli (*MPA*) & ejus coſinu, *s* & *c* reſpective, tum

$$EM^2 = s^2 \times \overline{EK+PM}^2 + \overline{c \times \overline{PM+EK}+KP}^2 \quad \&$$

$$FM^2 = s^2 \times \overline{FL+PM}^2 + \overline{c \times \overline{PM+FL}+LP}^2 \quad \text{pro}$$

quibus lineis ſubſtituantur earum valores, & reſultant duæ æquationes

$$v^2 = y^2 + \overline{2F - 2cE + 2cx} \times y + F^2 - 2cFE + E^2 + \overline{2cF - 2E}x + x^2$$

$$z^2 = y^2 + \overline{2H - 2cG + 2cx} \times y + H^2 - 2cHG + G^2 + \overline{2cH - 2G}x + x^2$$

ex his duabus æquationibus inveniantur valores incognitæ quantitatis (*x* & *y*) terminis vero incognitarum quantitatum (*z* & *y*,) quibus pro ſuis valoribus (*x* & *y*) datâ æquatione ſubſtitutis, reſultat æquatio relationem inter (*x* & *v*) *EM* & *FM* exprimens.

Sit vero angulus (*MPA*), quem ordinatæ (*PM*) cum ſuâ abſciſſâ *AP* faciunt, rectus, & coſinus (*c*) nihilo eſt æqualis, tranſeat etiam abſciſſa per puncta *E* & *F*, *i.e.* *F* & *H* nihilo ſint æquales, tum pro

x ſubſtituatur $\dfrac{z^2 - v^2 - G^2 + E^2}{2 \times \overline{E-G}}$ & pro $y^2 = v^2 - \overline{\dfrac{z^2 - v^2 - G^2 + E^2}{2 \times \overline{E-G}} - E}\,\Big|^2$

& reſultat æquatio quæſita.

COROLLARIUM.

Hinc conſtat dimenſiones æquationis relationem inter lineas *EM* *FM* exprimentis non plures eſſe dimenſionibus æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis quater ſumptis.

Et vice verſa non poſſunt dimenſiones æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, eſſe plures, dimenſionibus æquationis relationem inter lineas *EM* & *FM* exprimentis quater ſumptis.

Et

Et ſic de lineis ad plura puncta ductis &c. & proprietatibus exinde deducendis.

PROB. V.

Datâ æquatione relationem inter abſciſſam & ejus correſpondentes ordinatas exprimente, invenire æquationem relationem inter abſciſſam curvæ & ejus correſpondentes ordinatas exprimentem, in quâ abſciſſa & ejus ordinatæ datam algebraicam relationem habent ad abſciſſam datæ curvæ & ejus ordinatas.

Datis algebraicis æquationibus relationem inter abſciſſas & earum ordinatas datæ & requiſitæ curvæ exprimentibus, e reductione harum æquationum in unam, ita ut exterminentur omnes præter abſciſſam ejus correſpondentes requiſitæ curvæ ordinatas, vel problemate primo libri primi, ſequitur requiſita æquatio.

EXEMPLUM I.

Data æquatione, relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) definiens

$$y^n + Ay^{n-1} + By^{n-2} + Cy^{n-3} + \&c. = 0.$$

in qua coefficientes (A, B, C, D, &c.) ſunt reſpective abſciſſæ (x) quæcunque rationales functiones, invenire æquationem relationem definientem inter abſciſſam & correſpondentes ordinatas curvæ, quæ ita ſecat datæ curvæ ordinatas, ut ſummæ cuborum ex ſingulis ordinatis utrinque jacentibus, ſint inter ſe æquales.

Pro (y) ſubſtituatur $v + Q$ & reſultat æquatio

$$v^n + \overline{nQ + A} \times v^{n-1} + \overline{n \times \frac{n-1}{2} Q^2 + n-1 \times AQ + B} \times v^{n-2} + \overline{n \times \frac{n-1}{2} \times \frac{n-2}{3} Q^3 + \overline{n-1} \times \frac{n-2}{2} AQ^2 + \overline{n-2} BQ + C} \times v^{n-3} + \&c. = 0.$$

Summa cuborum ex ſingulis incognitæ quantitatis valoribus nihilo æqualis fiat, & per lemma primum libri primi reſultat æquatio

$$\overline{nQ + A}|^3 - 3 \times \overline{nQ + A} \times \overline{n \times \frac{n-1}{2} Q^2 + \overline{n-1} AQ + B} + 3 \times \overline{n \times \frac{n-1}{2} \times \frac{n-2}{3} Q^3 + \overline{n-1} \times \frac{n-2}{2} AQ^2 + \overline{n-2}\, BQ + C} = 0.$$

EXEMPLUM II.

Transformare unam curvam in alteram. Sint $(x, y, z, v;)$ duæ abſciſſæ & earum ordinatæ curvæ transformandæ & requiſitæ.

1mo Dentur tres æquationes relationem inter abſciſſam & ordinatas datæ curvæ, & abſciſſas & ordinatas datæ & requiſitæ curvæ, tres igitur ſunt æquationes, quatuor incognitas quantitates habentes, quibus in unam reductis, ita ut exterminentur incognitæ quantitates, reſultat æquatio relationem inter abſciſſam & ejus ordinatas requiſitæ curvæ exprimens.

Ex. gr. Sit data æquatio relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimens, & requiſitæ curvæ ſit abſciſſa (z) & ejus ordinata (y), tum ſit

$$x = \frac{pz+qv+r}{Az+Bv+C}$$

$$y = \frac{Pz+Qv+R}{Az+Bv+c}$$

quibus quantitatibus pro ſuis valoribus in datâ æquatione ſubſtitutis, reſultat æquatio relationem inter novam abſciſſam (z) & ejus ordinatis (v), exprimens, cujus dimenſiones æquales ſunt datæ æquationis dimenſionibus.

Data æquatione curvæ, invenire æquationem omnium curvarum datæ ſimilium.

Pro abſciſſâ & ordinatâ $(x$ & $y)$ ſubſtituantur reſpective nz & nv & fit.

Invenire, quando ordinatæ in una curvâ ſunt in datâ ratione ad ordinatas in alterâ.

Subſtituatur pro abſciſſâ & ordinatâ, mz & nv & fit.

Si geometricè requiritur hoc problema perficere, tum ſolummodo requiritur conſtructio.

2do Datis duabus æquationibus relationes inter duas abſciſſas & earum correſpondentes ordinatas exprimentibus, & tertiâ æquatione relationem exprimente inter abſciſſas & ordinatas duarum curvarum, invenire æquationem relationem inter duas vel tres ex incognitis quantitatibus, duabus abſciſſis & earum correſpondentibus ordinatis:

Hoc fit ex reductione æquationum in unam, ita ut incognitæ quantitates exterminentur.

COR.

COR. I.

E priori analyſi facile conſtant æquationum reſultantium dimenſiones, & conſequenter quot ſunt incognitarum quantitatum diverſi valores.

COR. II.

Illinc etiam deduci poteſt ſumma radicum, ſumma productorum ex ſingulis duabus in ſeſe ductis, & ſic deinceps.

PROB. VI.

Invenire algebraicarum curvarum proprietates, quæ pendent e dimenſionibus, quas in data quantitate habet quæcunque incognita quantitas (x), *vel ex æquationis dimenſionibus, cujus radices ſunt quæcunque incognita quantitas.*

Datis incognitæ quantitatis (x) dimenſionibus in datâ quantitate, exinde ſequitur e quot & qualibus factoribus in ſeſe ductis componitur data quantitas; & ſic de æquationibus.

EXEMPLUM.

Sit æquatio $y^n + \overline{a+bx}\, y^{n-1} + \overline{c+dx+ex^2}\, y^{n-2} + \&c. = 0$. relationem inter abſciſſam (x) & ejus correſpondentes ordinatas exprimens, & ejus ſubtangens (v) erit

$$\frac{ny^n + \overline{n-1} \times \overline{a+bx}\, y^{n-1} + \overline{n-2} \times \overline{c+dx+ex^2}\, y^{n-2} + \&c.}{by^{n-1} + \overline{d+2ex}\, y^{n-2} + \&c.}$$

Sint $\alpha, \beta, \gamma, \delta$, &c. ordinatæ (y) diverſi valores, quibus in poſteriori quantitate ſubſtitutis, reſultant ſubtangentes reſpective

$$\frac{n\alpha^n + \overline{n-1} \times \overline{a+bx} \times \alpha^{n-1} + \overline{n-2} \times \overline{c+dx+ex^2} \times \alpha^{n-2} + \&c.}{b\alpha^{n-1} + d + 2ex\alpha^{n-2} + \&c.}$$

$$\frac{n\beta^n + \overline{n-1} \times \overline{a+bx} \times \beta^{n-1} + \overline{n-2} \times \overline{c+dx+ex^2} \times \beta^{n-2} + \&c.}{b\beta^{n-1} + \overline{d+2ex}\, \beta^{n-2} + \&c.}$$

Ducantur omnes hi (n) diverſi ordinatæ valores in ſeſe, & aſcendit numerator ad $(n \times n)$ dimenſiones abſciſſæ (x), denominator vero aſſurgit ad $(n \times \overline{n-1})$ dimenſiones, ſupponatur & numerator & denominator nihilo reſpective æqualis, & ſint radices numeratoris & denominatoris reſpective $\pi, \varrho, \sigma, \tau$ &c. & $\zeta, \eta, \Theta, \iota$ &c. ſint (x) (& z) duo diverſi abſciſſæ valores, & A & B contentum ſub (n) ſubtangentibus ad abſciſſas $(x \,\&\, z)$ reſpective, tum

$$A : B :: \frac{\overline{x-\pi} \times \overline{x-\rho} \times \overline{x-\sigma} \times \overline{x-\tau} \, \&c.}{\overline{x-\zeta} \times \overline{x-n} \, \overline{x-\Theta} \times \overline{x-i} \, \&c.} : \frac{\overline{z-\pi} \times \overline{z-\rho} \times \overline{z-\sigma} \times \overline{z-\tau}}{\overline{z-\zeta} \times \overline{z-n} \times \overline{z-\Theta} \times \overline{z-i} \, \&c.}$$

COROLLARIUM.

Facile ex analyſeos principiis erui poteſt, quot caſus requiruntur ad probandam propoſitionem eſſe veram.

Et ſic de curvaturæ radiis & innumeris hujuſmodi propoſitionibus.

THEO.

Sint duæ curvæ, quarum quicunque termini ſint iidem in utrâque, tum omnis proprietas unius curvæ, quæ ex iis terminis pendet, i. e. *deduci poteſt, erit etiam alterius curvæ proprietas.*

COR. I.

Sit curva (n) dimenſionum, cujus crura in infinitum pergentia, (n) habent aſymptotes, & æquatio ad curvam & ejus crurum aſymptotes habent terminos (n & $n-1$) dimenſionum eoſdem, & conſequenter omnis proprietas curvæ, quæ ex iis terminis pendet, erit etiam proprietas aſymptotum.

COR. II.

Curvarum diametri pendent e terminis æquationis, in quibus dimenſiones abſciſſæ & ordinatæ inveniuntur maximæ (n), & proxime inferiores ($n-1$), & conſequenter diameter ad aſymptotes erit etiam diameter ad ipſam curvam.

COR. III.

Hinc ſequitur eandem habere inclinationem in datum diametrum aſymptotum ac curvæ ordinatæ.

COR. IV.

Sint ordinatæ ad abſciſſam utcunque inclinatæ, & earum ſummæ & ad aſymptotes & curvam erunt inter ſe æquales, & conſequenter nihilo æqualis eſt differentia inter ſummas ordinatarum ad aſymptotes & curvam, & exinde ſummæ ordinatarum partium inter curvæ crura & correſpondentes aſymptotes interceptarum utrinque diametri, erunt inter ſe æquales.

Centrum eſt interſectio duarum conjugatarum diametrorum.

Locus hujuſmodi punctorum eſt curva, quam dico centralem curvam.

Duæ

Duæ diametri ſe ipſos interſecent, & ad ſe invicem conſtanter tendant & ultimo coeant, tum locus punctorum hujuſmodi ultimarum inter ſectionum eſt curva, quam dico diametrialem curvam.

COR. V.

Sit curva (n) dimenſionum, (n) habens aſymptotes, & eandem & diametrialem & centralem curvam habent & aſymptotes & curva.

Diametri enim & curvæ & aſymptotum ſunt eædem, & exinde eædem ſunt & interſectiones conjugatarum diametrorum, & ultimæ diametrorum interſectiones, hæ vero interſectiones conſtituunt reſpective centralem & diametrialem curvam, & conſequenter hæ curvæ ſunt eædem.

Si curva non habet (n) aſymptotes, tum inveniri poſſunt aſymptotes & ellipſes, quarum æquationibus in ſeſe ductis, termini reſultantis æquationis, in quibus dimenſiones abſciſſæ & ordinatæ inveniuntur maximæ & proxime inferiores &c, iidem erunt ac datæ curvæ, & conſequenter eandem habent diametrum &c. aſymptotes & ellipſes conjunctim, ac data curva.

COR. VI.

Abſciſſa & ejus ordinatæ curvam interſecent in (n) punctis, & ratio, quam habet contentum ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ interceptis, ad contentum ſub ſingulis (n) ordinatis, pendet e terminis æquationis, in quibus inveniuntur maximæ & abſciſſæ & ordinatæ conjunctim dimenſiones; & conſequenter ſi curva habet (n) aſymptotes, & eadem linea eſt abſciſſa ad aſymptotes & curvam, & ad abſciſſam ordinatæ eandem habeant inclinationem, tum contentum ſub ſingulis (n) ordinatis curvæ, erit ad contentum ſub (n) ordinatis aſymptotum, in ratione contenti ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ curvæ interceptis, ad contentum ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ aſymptotum interceptis.

Et ſic de aſymptotibus & ellipſibus conjunctim.

COROLLARIUM VII.

Sint termini æquationis, in quibus maximæ invenientur abſciſſæ & ordinatæ $(x$ & $y)$ dimenſiones, contentum $\overline{y^2+x^2}\times\overline{y^2+x^2}\times\overline{y^2+x^2}$, &c. tum

tum contentum ſub (n) ordinatis ſemper erit æqualis contento ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ interceptis; & omnes ordinatæ cum ſuis diametris rectum angulum faciunt.

Et ſic de innumeris hujuſmodi proprietatibus.

PROB. VIII.

Fig. 10. *Sint deſcriptæ duæ curvæ, quarum æquationes relationem inter abſciſſas* ($x = AP$, $z = a\Pi$) *& earum reſpectivas correſpondentes ordinatas* ($y = PM$, $v = \Pi M$) *exprimentes ſunt*

$$Ay^n + By^{n-1} + Cy^{n-2} + Dy^{n-3} + \&c. = 0$$
$$av^m + bv^{m-1} + cv^{m-2} + dv^{m-3} + \&c. = 0$$

in quibus A, B, C, D, &c. a, b, c, d, &c. *reſpective denotant rationales abſciſſarum* (x & z) *functiones, invenire omnes æquationes, quas harum duarum curvarum ope conſtruere licet.*

Ita transformentur hæ æquationes, ut lineæ ($Ip = X$, $\gamma\pi = Z$) fiant abſciſſæ & ($pM = Y$, $\pi M = V$) fiant ordinatæ.

Scribantur pro lineâ AL (a), IL (b), ſinu anguli MpL (s), ſinu anguli MQp (c), & pro ſinu anguli pMQ quæ ſit $s\sqrt{1-c^2} + c\sqrt{1-s^2}$, ($S$), & ſinu anguli MPL (r), & pro ſinu anguli PLp quæ ſit $r\sqrt{1-c^2} + c\sqrt{1-r^2}$, ($C$) tum in æquatione

$$Ay^n + By^{n-1} + Cy^{n-2} + Dy^{n-3} + \&c. = 0$$

pro (x) ſcribatur $\frac{cX - SY}{r} + \frac{cb}{r} - a$, & pro ($y$), $\frac{CcX + \overline{rs - CS} \times Y}{rc} + \frac{Cb}{r}$.

Sic etiam transformari poteſt alia æquatio, ut ejus abſciſſa fiat ($\gamma\pi$) & ordinata (πM), quæ angulum facit cum ejus abſciſsâ ($\gamma\pi$) æqualem angulo (MpL), cujus ſinus eſt (s).

Pro abſciſſis (X, Z,) ſcribantur in utrâque reſultanti æquatione (t) & pro ordinatis (Y, V,) in utraque æquatione (w), reducantur duæ æquationes reſultantes in unam, ita ut exterminetur incognita quantitas (t), & reſultat æquatio, cujus radix eſt (w), & quæ continet in ſe omnes æquationes, quæ harum curvarum ope conſtrui poſſunt.

COROLLARIUM I.

Sint duæ curvæ datæ, & e prædictis conſtat æquationem generalem, quas earum ope conſtruere licet, non plures quam ſex incogni-

tas

tas quantitates habere, & conſequenter duarum datarum curvarum ope haud generaliter conſtrui poſſunt æquationes plures quam ſex dimenſiones habentes.

COROLLARIUM II.

Sit una curva, circulus, cujus haud datur radius, & non plures ſunt quam ſeptem incognitæ quantitates, & conſequenter ope circuli & datæ curvæ haud conſtrui poteſt curva plures quam ſeptem dimenſiones habens, & ſic denique de curvis plures incognitas quantitates habentibus.

COROLLARIUM III.

Datâ & æquatione conſtruendâ & unâ curvâ, cujus ope illam conſtruere licet, innumeræ aliæ inveniri poſſunt, quarum abſciſſæ & ordinatæ eædem ſunt ac abſciſſæ & ordinatæ datæ curvæ, & ordinatæ ad earum interſectionum puncta pertinentes erunt datæ æquationis radices.

Hæ datæ æquationes in quaſlibet quantitates ducantur & a ſeipſis ſubducantur, & æquatio reſultans erit quæſita æquatio.

COROLLARIUM IV.

Æquatio reſultans, cujus radix eſt (x) ſæpe habet poſſibiles radices, cum nulla datur poſſibilis interſectio, *i. e.* nullus poſſibilis valor ordinatæ, & hoc ſit ut conſtat e corollario ad problema xiii. libri primi quando duæ vel plures abſciſſæ valores ſunt inter ſe æquales.

COROLLARIUM V.

Æquatio (nm) dimenſionum conſtrui poteſt, duarum curvarum ope, quarum dimenſiones in ſeſe ductæ ad minimum æquant datæ æquationis (nm) dimenſiones.

COROLLARIUM VI.

Curva (n) dimenſionum duci poteſt per $\left(n \times \frac{n+3}{2}\right)$ puncta: ſint duæ curvæ (n) & (m) dimenſionum, & in curva (m) dimenſionum aſſumantur plura quam (nm) puncta, ſed pauciora quam $\left(n \times \frac{n+3}{2}\right)$, per quæ requiritur ducere curvam (n) dimenſionum, tum curva (n) dimenſionum reduci poteſt in duas vel plures, quarum una eſt altera curva (m) dimenſionum.

CO-

COROLLARIUM VII.

Sint $y^n + bxy^{n-1} + ex^2y^{n-2} + \&c.$

& $y^m + Bxy^{m-1} + Ex^2y^{m-2} + \&c.$

termini, in quibus maximæ inveniuntur æquationum incognitarum quantitatum (y & x) dimenſiones & communem habeant diviſorem, tum haud aſcendit æquatio reducta ad (nm) dimenſiones.

COROLLARIUM VIII.

Datis duabus æquationibus

$$py^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = 0$$

& $$Py^m + \overline{A+Bx}\,y^{m-1} + \overline{C+Dx+Ex^2}\,y^{m-2} + \&c. = 0$$

relationem inter abſciſſas & earum correſpondentes ordinatas duarum datarum curvarum exprimentibus, facile reduci poſſunt hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (x).

Interſectiones enim harum curvarum erunt quæſitæ æquationis radices, & datis æquationis radicibus, facile cognoſcitur æquatio ipſa.

Non opus eſt hic conſtruere æquationes. Facile conſtant infinitæ hujuſmodi propoſitiones.

PROB. VIII.

Fig. 11. *Datâ æquatione relationem inter abſciſſam* (AP) *& ejus correſpondentes ordinatas* (PM) *exprimente, invenire æquationem relationem inter abſciſſam* (Aπ) *& ejus ordinatas* (πm), *ad locum punctorum a quocunque modo e datâ æquatione deductorum.*

Datâ methodo inveniendi punctum (m) e datâ æquatione, datur etiam methodus inveniendi lineas ($A\pi$ & πm) abſciſſam & ordinatam quæſitæ curvæ, æquationes reſultantes ita reducantur in unam, ut exterminentur omnes præter duas ($A\pi$ & πm) abſciſſam & ordinatam quæſitæ curvæ, & fit.

EXEMPLUM.

Sit æquatio relationem inter abſciſſam ($AP = x$) & ejus correſpondentes ordinatas ($PM = y$) exprimens,

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} \ldots \overline{px^{n-1} + qx^{n-2} + rx^{n-3} + \&c.} \times y$$
$$+ Px^n + Qx^{n-1} + Rx^{n-2} + \&c. = 0.$$

Circa datum punctum (A) revolvatur recta (AM), quæ huic curvæ occurrat

occurrat in (n) punctis ($M, \overset{\prime}{M}, \overset{\prime\prime}{M}$, &c.) & ſi in câdem recta ſumatur ſemper Pn, ita ut

$$\frac{1}{Pn} = \frac{1}{PM} + \frac{1}{P\overset{\prime}{M}} + \frac{1}{P\overset{\prime\prime}{M}} + \&c.$$

tum erit locus puncti (n) recta.

Ita transformetur data æquatio, ut exprimat relationem inter abſciſſam AP, & ordinatam cum eâ angulum APL vel APN, &c. facientem, cujus ſinus eſt (σ), ſit etiam ſinus anguli (APM) rectus; tum in datâ æquatione pro (y) ſcribatur σv (PL, PN &c.), & pro x, $z\,(AP) + \sqrt{1-\sigma^2}\,v$; & termini reſultantis æquationis, in quibus invenitur nulla & una dimenſio ordinatæ (v), erunt reſpective

$$Pz^n + Qz^{n-1} + Rz^{n-2} + \&c.$$

$$\& \ \overline{p\sigma + nP\sqrt{1-\sigma^2}}\,z^{n-1} + \overline{q\sigma + \overline{n-1}\,Q\sqrt{1-\sigma^2}}\,z^{n-2} + \overline{r\sigma + \overline{n-2}\,R\sqrt{1-\sigma^2}}\,z^{n-3} + \&c.$$

ergo

$$\frac{1}{Pn} = \frac{1}{PM} + \frac{1}{P\overset{\prime}{M}} + \frac{1}{P\overset{\prime\prime}{M}} + \&c. =$$

$$\frac{\overline{p\sigma + nP\sqrt{1-\sigma^2}}\,z^{n-1} + \overline{q\sigma + \overline{n-1}\,Q\sqrt{1-\sigma^2}}\,z^{n-2} + \overline{r\sigma + \overline{n-2}\,R\sqrt{1-\sigma^2}}\,z^{n-3} + \&c.}{Pz^n + Qz^{n-1} + Rz^{n-2} + \&c.}$$

Sint $W\,(Pp)$ & $V\,(pn)$ abſciſſa & correſpondentes ordinatæ lineæ, cujus requiritur æquatio, tum $\frac{V}{\sigma} = Pn$ & $\frac{W}{\sqrt{1-\sigma^2}} = Pn$; quibus reductis, reſultat $Pn = \sqrt{W^2+V^2}$; & $\sigma = \frac{V}{\sqrt{W^2+V^2}}$, & $\sqrt{1-\sigma^2} = \frac{W}{\sqrt{W^2+V^2}}$; quibus quantitatibus pro ſuis valoribus in æquatione relationem inter Pn, z & σ exprimente, ſubſtitutis, reſultat æquatio relationem inter z, V & W exprimens

$$1 = \frac{\overline{pV + nPW}\,z^{n-1} + \overline{qV + \overline{n-1}\,QW}\,z^{n-2} + \overline{rV + \overline{n-2}\,RW}\,z^{n-3}}{Pz^n + Qz^{n-1} + Rz^{n-2} + \&c.}$$

ſed punctum (P) per hypotheſin datur, & conſequenter $AP = z$, ergo hæc æquatio relationem inter W & V abſciſſam & ordinatam lineæ requiſitæ deſignat, & quoniam in nullis terminis continentur plures quam una dimenſio abſciſſæ vel ordinatæ, linea requiſita erit recta.

COROLLARIUM.

Sit
$$\frac{1}{Pn} \text{ vel } \frac{1}{Pn^2} = \frac{1}{PM^2} + \frac{1}{\overset{\prime}{P}M^2} + \frac{1}{\overset{\prime\prime}{P}M^2} + \&c.$$
& locus punctorum (n) vel erit conica ſectio, vel duæ rectæ lineæ.

Sit
$$\frac{1}{Pn} \text{ vel } \frac{1}{Pn^3} = \frac{1}{PM^3} + \frac{1}{\overset{\prime}{P}M^3} + \frac{1}{\overset{\prime\prime}{P}M^3} + \&c.$$
& locus punctorum (n) vel erit curva ſecundi generis, vel conica ſectio & recta linea, vel tres rectæ lineæ, & ſic deinceps.

Et ſimiliter de infinitis aliis hujuſmodi propoſitionibus.

E ſubſtitutione præcedenti conſtant hæ propoſitiones.

EXEMPLUM II.

Fig. 12. Circa datum punctum (A) abſciſſæ verticem revolvatur recta linea AM, quæ occurrat algebraicæ lineæ cujuſcunque ordinis, in tot punctis, M, &c. quot ſunt ejus dimenſiones, & ſi in eâdem rectâ ſumatur AN, ita ut

$AN = AM + A\overset{\prime}{M} + A\overset{\prime\prime}{M} + \&c.$ tum locus punctorum (N) poteſt eſſe curva (n) dimenſionum, vel duæ vel plures curvæ, quarum dimenſionum ſumma eſt (n).

Sit æquatio relationem inter abſciſſam ($AP = x$) & ejus correſpondentes ordinatas (PM, y) exprimens,

$$Ay^n + \overline{a + bx}\, y^{n-1} + \overline{c + dx + ex^2}\, y^{n-2} + \overline{f + gx + hx^2 + kx^3} \times y^{n-3} \&c. = 0.$$

ita transformetur hæc æquatio; ut relationem inter abſciſſam (Ap) & ejus ordinatas (pM) exprimat; ſit angulus APM rectus, & pro ſinubus angulorum (ApP, PAp) ſcribantur s & $C = \sqrt{1 - s^2}$ reſpective, & pro lineis (Ap & pM) z & v, tum reſultant quantitatum (x) & (y) valores sz & $v + C(\sqrt{1-s^2})\, z$, quibus quantitatibus pro ſuis valoribus ſubſtitutis, coefficientes terminorum (z^n & z^{n-1}) reſultantis æquationis, erunt reſpective

$$AC^n + bC^{n-1}s + eC^{n-2}s^2 + kC^{n-3}s^3 + \&c.$$
& $$aC^{n-1} + dC^{n-2}s + hC^{n-3}s^2 + \&c.$$

& conſequenter ſumma radicum (z) erit

$$\frac{aC^{n-1} + dC^{n-2}s + hC^{n-3}s^2 + \&c.}{AC^n + bC^{n-1}s + eC^{n-2}s^2 + kC^{n-3}s^3 + \&c.} = AM + A\overset{\prime}{M} + A\overset{\prime\prime}{M} + \&c. = An;$$

ſint (V & w) reſpective Am & mn, abſciſſa & ejus ordinata curvæ quæſitæ

quæſitæ, tum eodem modo, quo in priori exemplo, colligi poteſt

$$An=\sqrt{w^2+V^2} \;\&\; s=\frac{V}{\sqrt{w^2+V^2}} \;\&\; C=\frac{w}{\sqrt{V^2+w^2}}$$

quibus quantitatibus pro ſuis valoribus in priori æquatione ſubſtitutis, reſultat æquatio relationem inter abſciſſam Am & ejus ordinatas (mn) quæſitæ curvæ exprimens

$$Aw^n+\overline{bV-a}\,w^{n-1}+\overline{eV^2-dV}\,w^{n-2}+\overline{kV^3-hV^2}\,w^{n-3}+\&c.=0,$$

cujus dimenſiones nonplures quam (n) eſſe poſſunt.

COROLLARIUM.

Ducatur quicunque parallelarum linearum numerus $(mn, \overset{1}{m}\overset{1}{n}, \&c.)$ Fig. 13. tranſeuntium per (n) hujuſmodi puncta, *i.e.* occurrentium curvæ quæſitæ in (n) punctis $(M, K, H, \&c.\ \overset{1}{M}, \overset{1}{K}, \overset{1}{H}, \&c.)$ & ſi in eadem recta ſemper ſumatur $(mo, \overset{1}{m}\overset{1}{o}, \&c.)$ ita ut

$$mo=mn+mK+mH+\&c.$$

$$\&\quad \overset{1}{m}\overset{1}{o}=\overset{1}{m}\overset{1}{n}+\overset{1}{m}\overset{1}{K}+\overset{1}{m}\overset{1}{H}+\&c.$$ erit locus punctorum $(o, \overset{1}{o}, \&c.)$ recta linea.

Et ſic de infinitis hujuſmodi propoſitionibus.

EXEMPLUM III.

Iiſdem poſitis, invenire curvam datæ curvæ centrorum locum.

Sit diameter, cujus abſciſſa (x) incipit a centro, & æquationis re- Fig. 14. lationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimentis, nihilo æquales erunt tres termini $(xy^{n-1}, y^{n-1}, x^{n-1}.)$

Ita transformetur data æquatio, ut exprimat relationem inter abſciſſam (Ap, z) & ejus ordinatas (pM, v); ſit angulus APM rectus, & pro ſinubus angulorum $(PAp$ & $ApM)$ ſcribantur reſpective s & c; & erunt ſinus angulorum $(prM$ & $rMp)$ reſpective, $\sqrt{1-s^2}$ & $sc+\sqrt{1-s^2}\sqrt{1-c^2}=C$, & conſequenter abſciſſa

$$AP=\sqrt{1-s^2}\,z-Cv \quad \&\text{ ordinata}$$

$$PM=sz+\frac{c-sC}{\sqrt{1-s^2}}\,v \quad \text{quibus}$$

quantitatibus pro ſuis valoribus in datâ æquatione ſubſtitutis; & in æquatione reſultanti

ſubſtituatur pro abſciſſa (z), $X+w$ (AN), & pro ordinatâ (y), $Y+u$ (Nm); & reſultant

$$AP=\sqrt{1-s^2}\,X-CY+\sqrt{1-s^2}\,w-Cu \ \&$$

$$PM=sX+\frac{c-Cs}{\sqrt{1-s^2}}Y+sw+\frac{c-Cs}{\sqrt{1-s^2}}u.$$

Subſtituantur hæ quantitates pro ſuis valoribus in datâ æquatione, & reſultantis æquationis terminorum (XY^{n-1}, Y^{n-1}, X^{n-1}) coefficientes erunt reſpective

$$nA\times\overline{\frac{c-Cs}{\sqrt{1-s^2}}}\Big|^{n-1}s+b\times\overline{\overline{\frac{c-Cs}{\sqrt{1-s^2}}}\Big|^{n-1}\sqrt{1-s^2}-\overline{n-1}\times\overline{\frac{c-Cs}{\sqrt{1-s^2}}}\Big|^{n-2}\times sC}$$

$+$ &c. . . . .

$$a\times\overline{\frac{c-Cs}{\sqrt{1-s^2}}}\Big|^{n-1}-d\times\overline{\frac{c-Cs}{\sqrt{1-s^2}}}\Big|^{n-2}C+b\times\overline{\frac{c-Cs}{\sqrt{-1s^2}}}\Big|^{n-3}C^2+\&c.$$

$$n\times A\,\overline{\frac{c-Cs}{\sqrt{1-s^2}}}\Big|^{n-1}\times\overline{sw+\frac{c-Cs}{\sqrt{1-s^2}}u}+b\times\&c.$$

. . . .

$$as^{n-1}+ds^{n-2}\sqrt{1-s^2}+bs^{n-3}\,\overline{1-s^2}+\&c.$$

$$+nAs^{n-1}\times\overline{sw+\frac{c-Cs}{\sqrt{1-s^2}}}+b\times\&c.$$

Fiant hæ tres coefficientes nihilo reſpective æquales, & pro coefficiente (C) ſubſtituatur ejus valor ($sc+\sqrt{1-s^2}\times\sqrt{1-c^2}$), & tres æquationes reſultantes ita reducantur in unam, ut exterminentur duæ incognitæ quantitates (c, C), & reſultat æquatio relationem inter abſciſſam (AN) & ordinatas (Nm) centralis curvæ quæſitæ exprimens.

Æquatio, relationem inter abſciſſam & ejus correſpondentes ordinatas curvæ exprimens, cujus locus eſt puncta interſectionum duarum diametrorum ſibi ipſis quam proximarum, e prædictis & incrementorum principiis facile erui poteſt.

PROB. IX.

Fig. 15. *Sit fluxio abſciſſæ* (AP=x) *invariabilis, & datâ algebraicâ æquatione relationem inter abſciſſam* (x) *& ejus correſpondentes ordinatas* (PM=y) *exprimente*

exprimente, invenire ſummam ex ſingulis valoribus fluxionalis quantitatis, quæ ſit quantitatum (x & y) *functio, cujus fluens inveniri poteſt.*

RESOLUTIO.

Inveniatur fluens, & e lemmatibus primo & ſecundo & eorum corollariis inveniatur ſumma ex ſingulis valoribus hujuſce fluentis, hujuſce ſummæ fluxio &c. erit ſumma quæſita.

EXEMPLUM I.

Datâ abſciſſâ, (x), & æquatione relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimente

$$y^n + \overline{a+bx}\, y^{n-1} + \overline{c+dx+ex^2}\, y^{n-2} + \&c. = 0.$$

ſummam ſubnormalium invenire.

Supponantur diverſi ordinatæ valores reſpective u, v, w, &c. & conſequenter diverſæ ſubnormales erunt reſpective $\frac{u\dot{u}}{\dot{x}}$, $\frac{v\dot{v}}{\dot{x}}$, $\frac{w\dot{w}}{\dot{x}}$, &c. ſit ($\dot{x}$) invariabilis quantitas, & harum quantitatum fluentes erunt $\frac{u^2}{2\dot{x}}$, $\frac{v^2}{2\dot{x}}$, $\frac{w^2}{2\dot{x}}$, &c.

Per lemma primum primi libri inveniri poteſt

$y^2 + v^2 + w^2 + \&c. = \overline{a+bx}^2 - 2 \times \overline{c+dx+ex^2}$, & conſequenter ſubnormalium ſumma quæſita erit

$$\frac{u\dot{u} + v\dot{v} + w\dot{w} + \&c.}{\dot{x}} = \frac{\overline{a+bx} \times b\dot{x} - \overline{d+2ex}\,\dot{x}}{\dot{x}} = ab - d + \overline{b^2 - 2e} \times x.$$

Si ordinatæ ſecant curvam in (n) punctis, & bis ſumma harum ſubnormalium nihilo æqualis eſt, tum vero, quæcunque ſit abſciſſa, nihilo æqualis erit.

Si curva habet generale centrum, & ejus abſciſſa per centrum tranſit, tum ſumma ſubnormalium erunt reſpective abſciſſis a centro incipientibus proportionales.

COR. I.

Hinc inveniri poſſunt innumeræ hujuſmodi propoſitiones. Aſſumatur enim quæcunque fluxionalis quantitas, abſciſſam, ordinatam &

& earum poteſtates & fluxiones involvens, ita vero ut ejus fluens inveniri poteſt.

Inveniantur lineæ, quæ datæ fluxionali quantitati ſunt æquales, deinde inveniatur ſumma ex ſingulis valoribus datæ fluxionalis quantitatis, & conſequenter invenitur ſumma ex ſingulis valoribus prædictarum linearum.

EXEMPLUM II.

Fig. 15. Aſſumatur fluxio $\frac{\ddot{y}}{\dot{x}^2}$, & ſit æquatio relationem inter abſciſſam ($AP = x$) & ejus correſpondentes ordinatas ($PM = y$)

$y^n + Ay^{n-1} + By^{n-2} +$ &c. $= o$, in quâ A & B &c. ſint rationales functiones abſciſſæ ($x = AP$).

Sit MQ chorda curvaturæ in directione (PM), & QR perpendiculum in tangentem demiſſum, & facile deduci poteſt $\frac{MQ}{QR^2} = \frac{\ddot{y}}{2\dot{x}^2}$.

Sed per vulgarem analyſin

$y + v + w +$ &c. $= A$ & conſequenter

$\frac{\ddot{y}}{\dot{x}^2} + \frac{\ddot{v}}{\dot{x}^2} + \frac{\ddot{w}}{\dot{x}^2} +$ &c. $= \frac{\ddot{A}}{\dot{x}^2}$ ergo ſumma quantitatum hujuſmodi $\frac{MQ}{QR^2}$ erit $\frac{\ddot{A}}{2\dot{x}^2}$.

EXEMPLUM.

Datâ eâdem æquatione

$y^n + Ay^{n-1} + By^{n-2} + Cy^{n-3}$ &c. $= o$.

invenire ſummam ex ſingulis valoribus ejus areæ.

Datâ abſciſâ (x), ſint $y, v, u, w,$ &c. diverſi ordinatæ (y) valores, &

$y + v + u + w +$ &c. $= A$ & conſequenter

$y\dot{x} + v\dot{x} + u\dot{x} + w\dot{x} +$ &c. $= A\dot{x}$, quarum fluentes ſunt areæ ipſæ, & earum ſumma quæſita æqualis eſt fluenti fluxionis ($A\dot{x}$).

COROLLARIUM

Nulla curva, quæ habet hyperbolicum crus conici generis, generaliter quadrari poteſt.

Ita transformetur enim data æquatio, ut reſultat æquatio relationem inter abſciſſam (quæ ſit hyperbolici cruris aſſymptoton) & ejus correſpondentes ordinatas exprimens; ex æquatione reſultante conſtat

ſtat ſummam ex ſingulis valoribus curvæ areæ i. e. fluentem fluxionis $(A\dot{x})$ haud generaliter quadrari poſſe.

Si vero curva ſit continua, eadem expreſſio vel æquatio, quæ generaliter quadrat unum curvæ crus, quadrabit etiam ſingula.

Et conſequenter ſi ſumma ex ſingulis valoribus areæ, haud inveniri poteſt, tum ab æquo ſequitur nullum curvæ crus quadrabile eſſe.

Et methodo haud multum diſſimili, ſequitur nullam eſſe algebraicam curvam, conſtantem ſolummodo ex ovalibus ſeſe in dato puncto vel punctis minime inter ſecantibus, quæ generaliter quadrari poteſt.

Ovales ſeſe ſecantes in datis punctis, nonnunquam generaliter quadrabiles eſſe docebit ſubſequens exemplum.

Sit æquatio curvæ

$$y = \overline{2a - 2x} \times \sqrt{2ax - x^2}, \text{ \&}$$

curva vel dici poteſt ovalis ſeſe interſecans in uno puncto, vel duæ ovales ſeſe interſecantes; ſed nullum habet crus in infinitum pergens.

Generaliter quadrari poteſt curva; & ejus area eſt in genere

$$\frac{2 \times \overline{2ax - x^2}^{1\frac{1}{2}}}{3}.$$

Sit curva algebraica, & conſtat ſummam ex ſingulis ejus areæ valoribus ſemper inveniri poſſe, conjunctim finitorum terminorum, circularium arcuum & logarithmorum ope.

Datâ abſciſsâ (x), & (n) & non plures ſunt diverſæ areæ ei correſpondentes, & conſequenter æquatio, cujus radix eſt area ipſa, (n) & non plures habere poteſt dimenſiones.

EXEMPLUM III.

Invenire ſummam ſolidorum a rotatione curvæ circa axem ſuum generatorum.

Summæ quæſitæ fluxio eſt $3{,}14159\text{\&c.} \times \overline{y^2 + v^2 + u^2 + w^2 + \text{\&c.}} \times \dot{x}$

$y^2 + v^2 + u^2 + w^2 + \text{\&c.} = A^2 - 2B$ & conſequenter $3{,}14159\ \overline{A^2 - 2B} \times \dot{x}$ eſt fluxio ejuſdem ſummæ, & ejus fluens ſumma ſolidorum quæſita.

EX-

EXEMPLUM IV.

Data abſciſſâ (x), invenire ſummam ex reciprocis ſubtangentibus.

Sit ultimus datæ æquationis terminus

$$px^n - qx^{n-1} + rx^{n-2} - sx^{n-3} + tx^{n-4} - \&c.$$

qui æqualis eſt contento ($y \times v \times u \times w \times$ &c.) ſub ſingulis ordinatæ (y) diverſis valoribus *i. e.*

$$yvuw\&c = px^n - qx^{n-1} + rx^{n-2} - sx^{n-3} + tx^{n-4} - \&c.$$

Inveniantur fluxiones ex utrâque æquationis parte, & oritur æquatio

$$\dot{y}vuw\&c + \dot{v}yuw\,\&c. + \dot{u}vyw\&c + \dot{w}yvu\,\&c. =$$
$$npx^{n-1} - \overline{n-1}\,qx^{n-2} + \overline{n-2}\,rx^{n-3} - \overline{n-3}\,sx^{n-4} + \&c.$$

dividatur prior æquationis pars per contentum ($yuvw$ &c.), poſterior vero per quantitatem $px^n - qx^{n-1} + rx^{n-2} - sx^{n-3} +$ &c. ei contento æqualem, & reſultat æquatio

$$\frac{\dot{y}}{y\dot{x}} + \frac{\dot{v}}{v\dot{x}} + \frac{\dot{u}}{u\dot{x}} + \frac{\dot{w}}{w\dot{x}} + \&c. = \frac{npx^{n-1} - \overline{n-1}\,qx^{n-2} + \overline{n-2}\,rx^{n-3} - \overline{n-3}\,sx^{n-4} + \&c.}{px^n - qx^{n-1} + rx^{n-2} - sx^{n-3} + \&c.}$$

Illæ autem quantitates $\frac{\dot{y}}{y\dot{x}}, \frac{\dot{v}}{v\dot{x}}, \frac{\dot{u}}{u\dot{x}}, \frac{\dot{w}}{w\dot{x}}$, &c. ſunt reſpective reciprocæ ſubtangentes, & earum ſumma quæſita eſt

$$\frac{npx^{n-1} - \overline{n-1}\,qx^{n-2} + \overline{n-2}\,rx^{n-3} - \overline{n-3}\,sx^{n-4} + \&c.}{px^n - qx^{n-1} + rx^{n-2} - sx^{n-3} + \&c.}$$

COROLLARIUM.

Fig.16. Occurrat recta quævis, per punctum datum (P) ducta, curvæ algebraicæ in tot punctis (A, B, C &c.), quot ſunt ejus dimenſiones: rectæ (AK, BL, CM, &c.) curvam tangentes in his punctis, ſecent abſciſſam *i. e.* alteram rectam lineam (PLM) per idem datum punctum ductâm, in punctis (K, L, M, &c.) & erit ſumma reciprocarum ſubtangentium $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} +$ &c. data quantitas.

Summa reciprocarum ſubtangentium ſolummodo pendet ex ultimo datæ æquationis termino,

$$px^n - qx^{n-1} - rx^{n-2} - sx^{n-3} + \&c.$$

abſciſſâ igitur eadem manente, in eam mutetur vero inclinatio ordinatarum, & a problematis primi corollario primo conſtat haud mutari ultimum datæ æquationis terminum, & conſequenter ſi ordinatæ diverſas haben-

habentes ad abſciſſam inclinationes, per idem punctum (P) abſciſſæ tranſeant *i. e.* habeant eoſdem quantitatis (x) valores, & erunt ſummæ reciprocarum ſubtangentium reſpective inter ſe æquales.

PROB. X.

Data æquatione, relationem inter abſciſſam (x) *& ejus ordinatas* (y) *exprimente, & fluxionali quantitate, quæ eſt functio abſciſſæ* (x) *& ordinatæ* (y), *cujus fluens haud inveniri poteſt, invenire ſummam ex ſingulis hujus functionis valoribus.*

Ex datâ æquatione inveniatur prima fluxio ordinatæ (y), & ſic ſecunda, tertia, &c. fluxiones ordinatæ, quibus valoribus in datâ fluxionali quantitate ſubſtitutis, reſultat algebraica quantitas; cujus ſumma ex ſingulis valoribus erui poteſt ex libri primi lemmate primo, & ſecundo & ejus corollariis.

EXEMPLUM I.

Datâ æquatione curvæ $y^2 - Ay + B = 0$, in quâ A & B ſunt quæcunque abſciſſæ (x) functiones, invenire æquationem cujus radices ſunt ordinatæ (y) fluxiones.

Sint duæ ordinatæ (y) radices reſpective y & v, & ſumma requiſitæ æquationis radicum, erit $\dot{y} + \dot{v} = \dot{A}$; & productum ſub binis valoribus contentum, erit fractio

$$\dot{y}\dot{v} = \frac{\dot{A}y - \dot{B}}{2y - A} \times \frac{\dot{A}v - \dot{B}}{2v - A};$$

cujus denominator eſt

$$4yv - 2A \times \overline{y + v} + A^2 = 4B - A^2,$$

Numerator vero

$$\dot{A}^2 yv - \dot{A}\dot{B} \times \overline{y + v} + \dot{B}^2 = B\dot{A}^2 - A \times \dot{A}\dot{B} + \dot{B}^2$$

& conſequenter æquatio quæſita eſt

$$\dot{y}^2 - \dot{A}\dot{y} + \frac{B\dot{A}^2 - A\dot{A}\dot{B} + \dot{B}^2}{4B - A^2} = 0$$

EXEMPLUM II.

Invenire æquationem, cujus radices ſunt ſubtangentes curvæ, cujus æquatio eſt $y^2 - Ay + B = 0$.

Sint y & v duo ordinatæ (y) valores, & duæ ſubtangentes erunt $\frac{\overline{2y^2 - Ay} \times \dot x}{\dot A y - \dot B}$ & $\frac{\overline{2v^2 - Av} \times \dot x}{\dot A v - \dot B}$; quibus fractionibus ad communem denominatorem reductis, fit denominator

$$\dot A^2 yv - \dot A \dot B \times \overline{y + v} + \dot B^2 = B \dot A^2 - A \times \dot A \dot B + \dot B^2;$$

Numerator vero

$$\overline{\dot x \times 2 \dot A yv \times \overline{y + v} - 2 A \dot A yv - 2 \dot B\, \overline{y^2 + v^2} + A \dot B \times \overline{y + v}} =$$

$$= \overline{\dot x \times 2 B A \dot A - 2 B A \dot A - 2\, \overline{A^2 - 2B} \times \dot B} + A^2 \dot B,$$ & conſequenter ſumma ſubangentium erit

$$\frac{4 B \dot B - A^2 \dot B}{\dot A^2 B - \dot B \dot A A + \dot B^2}\, \dot x.$$

Fractionis, quæ ſit productum ſub duabus ſubtangentibus contentum, numerator erit

$$\dot x^2 \times \overline{4 y^2 v^2 - 2 A yv \times \overline{y + v} + A^2 \times yv} =$$

$\dot x^2 \times \overline{4 B^2 - 2 A^2 B + A^2 \times B} = 4 B^2 - A^2 B$; & conſequenter æquatio, cujus radices ſunt ſubtangentes (w), erit

$$w^2 - \frac{\overline{4 B \dot B - A^2 \dot B} \times \dot x}{\dot A^2 B - \dot B \dot A A + \dot B^2}\, w + \frac{\overline{4 B^2 - A^2 B} \times \dot x^2}{\dot A^2 B - \dot B \dot A A + \dot B^2} = 0.$$

EXEMPLUM III.

Datâ æquatione curvæ $y^3 + By - C = 0$, in quâ (B) & (C) ſunt quæcunque abſciſſæ functiones, invenire æquationem, cujus radices ſunt ordinatæ (y) fluxiones.

Sint tres radices ordinatæ (y, v, w), tum ſumma earum fluxionum $\dot y + \dot v + \dot w = 0$. &

$$\dot y \dot v + \dot v \dot w + \dot w \dot v = \frac{\dot C - \dot B y}{3y^2 + B} \times \frac{\dot C - \dot B v}{3v^2 + B} + \frac{\dot C - \dot B y}{3y^2 + B} \times \frac{\dot C - \dot B w}{3w^2 + B} +$$

$\frac{\dot C - \dot B v}{3v^2 + B} \times \frac{\dot C - \dot B w}{3w^2 + B}$; reducantur hæ fractiones in communem denominatorem,

minatorem, & erit denominator
$\overline{3y^2+B}\times\overline{3v^2+B}\times\overline{3w^2+B}=27C^2+4B^3$; numerator vero
$\overline{3w^2+B}\times\overline{\dot{C}-\dot{B}y}\times\overline{\dot{C}-\dot{B}v}+\overline{3v^2+B}\times\overline{\dot{C}-\dot{B}y}\times\overline{\dot{C}-\dot{B}w}+\overline{3y^2+B}\times\overline{\dot{C}-\dot{B}v}\times\overline{\dot{C}-\dot{B}w}=$
$\dot{B}^2B^2-3B\dot{C}^2+9C\dot{B}\dot{C}$, ergo ſumma productorum ex quibuſque binis in ſeſe ductis, erit $\frac{\dot{B}^2\times B^2-3\dot{C}^2B+9C\dot{B}\dot{C}}{27C^2+4B^3}$; & contentum ſub tribus radicibus $(\dot{y}\dot{v}\dot{w})$ erit $\frac{\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B}{27C^2+4B^3}$; unde æquatio quæſita eſt $\dot{y}^3+\frac{B^2\dot{B}^2-3\dot{C}^2B+9C\dot{B}\dot{C}}{27C^2+4B^3}\dot{y}-\frac{\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B}{27C^2+4B^3}=0$.

COROLLARIUM.

Eodem modo deduci poteſt æquatio, cujus radices ſunt ſecundæ datæ æquationis ordinatæ (y) fluxiones, ex æquatione, cujus radices ſunt primæ datæ æquationis fluxiones, quo deducitur ea æquatio ex datâ; & ſic deinceps.

EXEMPLUM IV.

Invenire ſubtangentium curvæ, cujus æquatio eſt $y^3+By-C=0$, ſummam.

Sint (y, v, w) diverſi ordinatæ (y) valores, & ſubtangentes erunt reſpective

$$\frac{\overline{3y^3+By}\times\dot{x}}{\dot{C}-\dot{B}y},\quad\frac{\overline{3v^3+Bv}\times\dot{x}}{\dot{C}-\dot{B}v},\quad\frac{\overline{3w^3+Bw}\times\dot{x}}{\dot{C}-\dot{B}w}$$

quibus fractionibus ad communem denominatorem reductis, fit denominator.

$$-\dot{B}^3yvw+\dot{B}^2\dot{C}\times\overline{yv+yw+vw}-\dot{B}\dot{C}^2\times\overline{y+v+w}+\dot{C}^3$$

$=\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B$; & eâdem methodo invenitur numerator
$\overline{4B^2\dot{C}\dot{B}-5BC\dot{B}^2+9\dot{C}C^2}\times\dot{x}$, & conſequenter ſubtangentium ſumma quæſita eſt

$$\frac{\overline{-5\dot{B}^2CB+4\dot{B}CB^2+9\dot{C}^2C}\times\dot{x}}{\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B}.$$

His principiis poſitis, inveniri poſſunt infinitæ conſimiles curvarum proprietates.

Hoc problema multis in caſibus ſolvi poteſt e transformatione duarum vel plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

Hinc facile conſtat numerum perpendiculorum a dato puncto ad curvam (n) dimenſionum ductorum, non majorem eſſe quam n^2, & numerum tangentium a dato puncto ad curvam ductarum etiam non majorem eſſe quam (n^2).

Sæpe vero ſolutionem his plane diverſam, e lemmate primi libri ſecundo facile excudat aliquis.

PROB. XI.

Sint (n) *rectæ* (A B, C D &c.), n *etiam aliæ* (E F, G H &c.), n *aliæ* (LN, OP &c.) *& ſic deinceps; & a quocunque puncto* (M) *ad has rectas ducantur rectæ* Mα, Mβ, &c. Mε, Mζ, &c. Mλ, Mμ, Mν, &c. *in datis angulis vel directionibus, ſingulæ ad ſingulas; & ducatur contentum ſub* (n) *lineis* M$\alpha \times$ M$\beta \times$ &c. & M$\varepsilon \times$ M$\zeta \times$ &c. M$\lambda \times$ M$\mu \times$ &c. *reſpective in quaſlibet datas quantitates; quantitatum reſultantium ſumma nihilo æqualis fiat & locus punctorum* (M), *vel erit curva* (n) *dimenſionum, vel erunt curvæ vel rectæ, quarum dimenſiones ſimul ſumptæ erunt* (n).

Fig. 16. Aſſumatur enim recta (AP) pro abſciſſâ & PM ordinata (y), tum

$$M\alpha = \alpha y + \beta x + \gamma,\quad M\varepsilon = \varepsilon y + \zeta x + n,\quad M\lambda = \lambda y + \mu x + \nu$$
$$M\beta = \overset{1}{\alpha} y + \overset{1}{\beta} x + \overset{1}{\gamma},\quad M\zeta = \overset{1}{\varepsilon} y + \overset{1}{\zeta} x + \overset{1}{n},\quad M\mu = \overset{1}{\lambda} y + \overset{1}{\mu} x + \overset{1}{\nu}$$
&c. &c. &c. &c. &c. &c.

Ducantur hæ quantitates ($M\alpha$, $M\beta$, &c. & $M\varepsilon$, $M\zeta$, &c. $M\lambda$, $M\mu$, &c.) reſpective in ſeſe; & contenta multiplicentur reſpective in datas quantitates, fiant quantitates reſultantes inter ſe æquales, & reſultat æquatio relationem inter abſciſſam $(AP = x)$ & ejus correſpondentes ordinatas (PM, y) exprimens, (n) dimenſiones habens.

EXEMPLUM I.

Fig. 17. Sit Trapezium $(ABCD)$, & a puncto (M) ad ejus latera in datis angulis ducantur lineæ MQ, MR, MS, MT, ſingulæ ad ſingula; & ſit

ſit rectangulum ductarum ad oppoſita duo latera $MQ \times MS$ in datâ ratione $(m : 1)$ ad rectangulum ductarum ad alia duo latera $MR \times MT$; tum punctum (M) vel erit punctum ad conicam ſectionem, vel ad duas rectas lineas.

Aſſumatur enim pro abſciſſa $AP\,(x)$, & pro (PM) ordinata (y), tum

$$MS = \alpha y + \beta x + \gamma \quad \& \quad MR = \delta y + \epsilon x + \zeta$$

$$\times \quad MQ = y \qquad\qquad MT = ny + \Theta x$$

$$MQ \times MS = \alpha y^2 + \beta xy + \gamma y = m \times MR \times MT = m\delta n y^2 + \overline{\delta\Theta + \epsilon n} \times x + n\zeta \times y + \Theta\epsilon x^2 + \zeta\Theta x,$$

quæ eſt æquatio ad conicam ſectionem, vel duas rectas lineas.

COROLLARIUM I.

Hinc conſtat curvam tranſire per angulum Trapezii (A), quoniam ſi abſciſſa (x) nihilo eſt æqualis, tum correſpondens ordinata (y) nihilo eſt æqualis, & ſic probari poteſt conicam ſectionem tranſire per puncta (B, C, D), & conſequenter Trapezium in conicâ ſectione inſcriptum eſſe.

COROLLARIUM II.

Sit conica ſectio, in quâ inſcribitur Trapezium $(ABCD)$, & M quodlibet peripheriæ punctum; in lineas AB, BC, CD, DA in datis angulis ducantur MQ, MR, MS, MT; & erit rectangulum $MQ \times MS$ ad rectangulum $MR \times MT$ in datâ ratione.

EXEMPLUM II.

Sint tres lineæ AB, CD, EF; & tres aliæ BG, DH, FI, & a puncto (M) in datis angulis ad lineas prædictâs ducantur lineæ MQ, MR, MS, MV, MW, MZ reſpective; & ſit ſolidum $MQ \times MR \times MS$ ad ſolidum $MV \times MW \times MZ$ in datâ ratione; & punctum (M) ponitur ad curvam ſecundi generis. Fig. 18.

Hoc demonſtrari poteſt eodem modo, quo prior exemplum.

COROLLARIUM I.

Puncta B, D, F, L, N &c. in quibus lineæ AB, CD, EF ſecant lineas, BG, DH, FI, erunt puncta ad curvam ſecundi generis.

COROLLARIUM II.

Sit curva ſecundi generis, & ducantur tres lineæ AB, CD, EF, & tres aliæ BG, DH, FI, has tres ſecantes in punctis curvæ $(B, D, F,$

$L,$

L, *N*, &c.) & a quolibet curvæ puncto in hæc latera ducantur lineæ in datis angulis MQ, MR, MS, MZ, MV, MW, reſpective; tum ſolidum $MQ \times MR \times MS$ erit ad ſolidum $MZ \times MV \times MW$ in datâ ratione.

Et ſic in quibuſdam caſibus de lineis quarti, quinti, ſexti, &c. ordinum.

Fig. 16. Sit curva ($m\overset{\prime}{m}\overset{\prime\prime}{m}$ &c.) habens (m) dimenſiones, ſint etiam ($n-m$) rectæ (AB, CD &c.) & (n) etiam rectæ (EF, GH &c.); a quocuque puncto (M) in datâ directione (MP) ad curvam ducatur linea $Mm\overset{\prime}{m}\overset{\prime\prime}{m}$ &c. eam ſecans in (m) punctis (m, $\overset{\prime}{m}$, $\overset{\prime\prime}{m}$, &c.); & ſic agantur in datis angulis ad rectas (AB, CD &c, EF, GH &c.) lineæ $M\alpha$, $M\beta$, $M\gamma$, &c, $M\epsilon$ $M\zeta$, Mn &c. & ſit contentum ſub (n) lineis $Mm \times M\overset{\prime}{m} \times M\overset{\prime\prime}{m} \times$ &c. $\times$ $M\alpha \times M\beta \times M\gamma$ ad contentum ſub (n) lineis $M\epsilon \times M\zeta \times Mn \times M\theta$ &c. in datâ ratione; tum punctum (M) erit punctum ad curvam (n) dimenſionum, vel ad curvas, quarum dimenſiones ſimul ſumptæ ſunt (n).

Aſſumatur enim recta (AP, x) pro abſciſſâ, & PM pro ordinata (y), ſit etiam æquatio relationem inter abſciſſam ($AP = x$) & datæ curvæ ordinatas (v) Pm, $P\overset{\prime}{m}$ &c. ubi m & $\overset{\prime}{m}$ in eâdem directione, *i.e.* linea cum M jacent, exprimens

$Av^m + \overline{a+bx}\,v^{m-1} + \overline{c+dx+ex^2}\,v^{m-2} + \&c. = o$, & contentum ſub ($m$) lineis $Mm \times M\overset{\prime}{m} \times M\overset{\prime\prime}{m} \times$ &c. erit

$$Ay^m + \overline{a+bx}\,y^{m-1} + \overline{c+dx+ex^2}\,y^{m-2} + \&c.$$

lineæ vero $M\alpha$, $M\beta$, $M\gamma$, &c. ſint

$M\alpha = \alpha y + \beta x + \gamma$ & ſic de lineis $M\epsilon$, $M\zeta$, Mn, &c.

$M\beta = \overset{\prime}{\alpha} y + \overset{\prime}{\beta} x + \overset{\prime}{\gamma}$ &c.

Mn, &c. tum contentum ſub (m) lineis ($Mm \times M\overset{\prime}{m} \times M\overset{\prime\prime}{m} \times$ & $(\overline{n-m})$ $M\alpha$, $M\beta$, $M\gamma$ &c. *i.e.*

$$\overline{Ay^m + \overline{a+bx}\,y^{m-1} + \overline{c+dx+ex^2}\,y^{m-2} + \&c.} \times \overline{\alpha y + \beta x + \gamma} \times \overline{\overset{\prime}{\alpha} y + \overset{\prime}{\beta} x + \overset{\prime}{\gamma}}$$

erit in datâ ratione ad contentum ſub (n) lineis $M\epsilon \times M\zeta \times Mn \times M\theta$ &c. cujus proportionis extremis & mediis in ſeſe ductis, & inter ſe æquales eſſe ſuppoſitis, reſultat æquatio, in quâ (n) & non plures inveniuntur

untur dimenſiones incognitarum quantitatum (x & y) abſciſſæ & ordinatæ quæſitæ.

Sint plures curvæ & a quocunque puncto in eas ducantur lineæ in datis directionibus, & eadem erit ratiocinandi methodus de hujuſmodi punctorum loco.

Hinc facile multæ hujuſmodi proprietates deduci poſſunt, quæ omnibus algebraicis curvis competunt.

COROLLARIUM.

Sint quæcunque lineæ, quæ exprimantur terminis abſciſſæ & ordinatæ (x & y), &c. e priori analyſi inveniri poteſt, an data æquatio

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = 0$$

eſt integra harum linearum functio.

Ex. gr. Curvæ parium dimenſionum ſemper formari poſſunt e linearum in quæcunque puncta ductarum functionibus; nullam enim habet quadratum lineæ a dato puncto ad curvam ductæ, irrationalitatem: et infinitis modis variari poſſunt hæ functiones.

Hinc deduci poſſunt quam plurimæ conicarum ſectionum proprietates.

PROB. XII.

Deſcribere curvas, quarum æquationes relationem inter abſciſſam & ejus correſpondentes ordinatas dantur.

Inveniatur methodus conſtruendi generaliter datam æquationem, & fit.

COROLLARIUM I.

Hinc omnes curvæ trium & quatuor dimenſionum deſcribi poſſunt, datur enim methodus in genere conſtruendi æquationes trium & quatuor dimenſionum, circuli & conicæ parabolæ ope; datur igitur methodus deſcribendi curvas trium vel quatuor dimenſionum.

COROLLARIUM II.

Multæ ſunt curvæ ſuperiorum dimenſionum, quæ quarumcunque, duarum curvarum inferiorum ordinum ope, haud generaliter deſcribi poſſunt.

Æquationes enim, quas conſtruere licet harum duarum curvarum ope, per octavi problematis corollarium primum libri ſecundi, haud ſæpe includunt omnes, quicunque ſit valor abſciſſæ, harum curvarum æquationes.

Hinc ſequitur in deſcribendis curvis (n) dimenſionum, haud raro opus eſſe quam plurimis inferiorum ordinum curvis.

CO-

COROLLARIUM III.

Si recta rotetur, ſemper tranſeuns per datum curvæ punctum, tum dimenſiones æquationis conſtruendæ ſolummodo erunt $(n-1)$.

Si per duplex punctum curvæ tranſit, tum dimenſiones ſolummodo erunt $(n-2)$; & ſic deinceps.

COROLLARIUM IV.

Curvam (n) dimenſionum, cujus $(n-1)$ crura ſeſe interſecant in dato puncto (A), deſcribere.

Fig. 20. Sit æquatio relationem inter abſciſſam (AP, x) a puncto (A) incipientem & ejus ordinatas (PM, y), in eam ad rectum angulum inſiſtentes

$$Ay^n + \overline{a+bx}\, y^{n-1} + \overline{c+dx+ex^2}\, y^{n-2} + \&c. = 0.$$

Ita transformetur hæc æquatio, ut exprimat relationem inter abſciſſam Ap, z & ordinatam (pM, v).

Sint q & s ſinus & coſinus anguli ApP; tum ſubſtituatur in datâ æquatione pro x, qz; & pro y, $v+sz$; & coefficientes terminorum (z^n, z^{n-1}) erunt reſpective

$$As^n + bs^{n-1}q + es^{n-2}q^2 + \&c.$$

$$\& \quad as^{n-1} + ds^{n-2}q + hs^{n-3}q^2 + \&c.$$

& conſequenter ſi in abſciſſâ (Ap) ſumatur linea $\left(Am = \dfrac{as^{n-1} + ds^{n-2}q + hs^{n-3}q^2 + \&c.}{As^n + bs^{n-1}q + es^{n-2}q^2 + \&c.}\right)$, erit (m) punctum in peripheriâ curvæ.

Hinc facile inveniri poteſt quilibet punctorum numerus, per quæ tranſit curva.

Hujus quantitatis e geometriæ principiis ſequitur conſtructio.

Si curva habet punctum, in quo $(n-2)$ crura ſeſe interſecant, tum inveniri poſſunt puncta, per quæ tranſit curva, quadraticæ æquationis ope, unde facile conſtrui poteſt curva. Et ſic deinceps.

Non opus eſt hic deſcribere curvas motu datorum angulorum vel curvarum circa data puncta, & ſic deinceps: e generalibus principiis facile deduci poſſunt.

Datâ methodo deſcriptionis, erui poſſunt curvæ dimenſiones, vel e numero punctorum, in quibus a rectâ lineâ ſecari poteſt; vel e dimenſionibus æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas relationem exprimentis.

Ex.

Ex. gr. Sint (n) puncta (a, b, c, d, &c.) & ſit (m) punctum, a quo ad data puncta (a, b, c, d, &c.) ducuntur lineæ (am, bm, cm, &c.) & ſit ſumma linearum $am + bm + cm + dm +$ &c. data quantitas; & locus punctorum (m) nonpoteſt eſſe curva plurium quam (2^n) dimenſionum. Fig. 21.

Datis quibuſdam curvis, ex iis ſæpe deduci poſſunt aliæ.

E punctis datæ curvæ, quolibet modo deriventur alia puncta; & conſequenter e datâ curvâ ſequitur altera.

Ex. gr. Sit curva MNS, & a dato puncto C in tangentem curvæ ſemper demittatur perpendiculum (CP), tum locus hujuſmodi punctorum (P) erit deducta curva. Fig. 22.

Et ſic ducatur a puncto (c) perpendiculum ad lineam CS, jungentem punctum curvæ (S) & datum punctum (C); hoc vero perpendiculum incidat in lineam (CS) in puncto (n); & locus hujuſmodi punctorum erit nova curva. Fig. 22.

Et ſic a dato puncto (C) ducatur perpendiculum in curvæ perpendiculum, & formatur alia curva: ſed de his ſatis. Fig. 23.

De curvis per data puncta tranſeuntibus, vel datas lineas tangentibus.

1. Sit curva data, tum tria puncta, vel tres tangentes, &c. ſufficiunt ad determinandam ejus poſitionem.

2. Numerus tangentium vel punctorum, quæ ſufficiunt ad determinandam curvam, pendet e numero incognitarum coefficientium, quas habet data æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens.

Ex. gr. Si curva (n) dimenſionum generale habet centrum, quod ſit (C), duci poteſt curva per $n \times \frac{n+1}{2}$ puncta; tot enim habet æquatio incognitas coefficientes.

Data æquationis formula, relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, invenire curvam, quæ tranſeuns per data puncta, tangit datas lineas, &c.

Aſſumatur linea (AP) pro abſciſsâ, alia vero (PM) pro ordinatarum directione; pro ſingulo dato puncto, datur & abſciſſa & correſpondens ordinata, quibus pro ſuis valoribus in datâ æquatione ſubſtitutis, reſultant tot æquationes, quot dantur puncta. Fig. 24.

 Ducatur

Ducatur linea (nm) parallela ordinatæ (MP), ſecans tangentem (EF) in (n), e datâ æquatione inveniatur ratio, quam habent fluxiones abſciſſæ & ordinatæ, quæ eſt $Fm : nm$; inveniatur etiam ſubtangens, cui æqualis eſt $x+AF$; ita transformentur omnes hæ æquationes in unam, ut exterminentur omnes præter unam incognitam coefficientem, & reſultat æquatio, cujus incognita quantitas eſt ea coefficiens.

Fortaſſe vero requirit problema, ut prius transformetur abſciſſa in alteram, quod fit per problema primum.

E prædictis æquationibus conſtant dimenſiones reſultantis æquationis.

THEO.

Fig. 25. *Sit circulus* ABDEFG &c. P, *cujus centrum eſt* C, *& radius, unitas: ſint etiam arcus* AB, BD, DE, EF, FG, *&c. inter ſe æquales, pro chordâ circuli* (PB) *ſcribatur* (p), *& aſſumatur quadratica æquatio* $x^2 - px + 1 = 0$, *ſupponantur duæ hujuſce æquationis radices* α *&* β; *tum erunt reſpective chordæ*

$$BP = \alpha + \beta$$
$$PD = \alpha^2 + \beta^2$$
$$PE = \alpha^3 + \beta^3$$
$$PF = \alpha^4 + \beta^4 \quad \text{\& ſic deinceps.}$$

DEMONSTRATIO.

Sit $BP = p = a$. tum e trigonometriâ ſequitur

$$PD = pa - 2 = b, \ \&$$
$$PE = pb - a = c,$$
$$PF = pc - b = d. \ \text{\& ſic deinceps.}$$

Sit quadratica æquatio, $x^2 - px + 1 = 0$, tum per *Newtoni* regulam,

$$\alpha + \beta = p = BP$$
$$\alpha^2 + \beta^2 = pa - 2 = PD$$
$$\alpha^3 + \beta^3 = pb - a = PE$$
$$\alpha^4 + \beta^4 = pc - b = PF. \ \text{\& ſic deinceps.}$$

COROLLARIUM I.

Sit arcus $AG = n \times AB$, & chorda $BP = p$; ſit etiam quadratica æquatio ($x^2 - px + 1 = 0$), cujus radices ſunt α & β; tum chorda ($PG = \alpha^n + \beta^n$) erit per lemma primum præcedentis libri

$$p^n - np^{n-2} + n \times \frac{\overline{n-3}}{2} p^{n-4} - n \times \frac{\overline{n-5}}{2} \times \frac{\overline{n-4}}{3} p^{n-6} + n \times \frac{\overline{n-7}}{2} \times \frac{\overline{n-6}}{3} \times \frac{\overline{n-5}}{4} p^{n-8} + \&c.$$

CO-

COROLLARIUM II.

Duæ radices (α & β) quadraticæ æquationis $x^2-px+1=0$ ſunt reſpective $\sqrt{\frac{1}{4}p^2-1}+\frac{1}{2}p$ & $\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-1}$, & conſequenter $\overline{\frac{1}{2}p+\sqrt{\frac{1}{4}p^2-1}}^n+\overline{\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-1}}^n=\alpha^n+\beta^n=p^n-np^{n-2}+n\times\frac{n-3}{2}p^{n-4}-n\times\frac{n-5}{2}\times\frac{n-4}{3}p^{n-6}+\&c.=0$.

COROLLARIUM III.

Hinc etiam reduci poteſt in ſuos quadraticos diviſores, quantitas ($\pi^{2n}-\alpha\pi^n+1=0$.)

Sit quadratica æquatio $x^2-\overline{\frac{\pi^2+1}{\pi}}\,x+1=0$, cujus radices erunt reſpective π & $\frac{1}{\pi}$; & aſſumantur duo arcus AB & AG, quorum AG ſit $n\times AB$; ſit etiam chorda $PG=\alpha$, $PB=\frac{\pi^2+1}{\pi}$, & radius unitas; tum erit $PG=\alpha=\pi^n+\frac{1}{\pi^n}$, & conſequenter $\pi^{2n}-\alpha\pi^n+1=0$. Datâ chordâ (α), invenitur arcus (PG), & ejus ſupplementum (AG); & deinde arcus $AB=\frac{AG}{n}$, vel $\frac{360+AG}{n}$, vel $\frac{2\times 360+AG}{n}$, vel $\frac{3\times 360+AG}{n}$, vel $\frac{4\times 360+AG}{n}$, & ſic deinceps, dato igitur arcu (AB) invenitur ejus ſupplementi chorda (PB), pro quâ ſcribatur β, & erit $\frac{\pi^2+1}{\pi}=\beta$, quâ æquatione reductâ, reſultat quadraticus diviſor $\pi^2-\beta\pi+1=0$.

COROLLARIUM IV.

Cubica æquatio ($\pi^{3n}-\alpha\pi^{2n}+\beta\pi^n-1=0$) in ſimplicem & quadraticam; biquadratica ($\pi^{4n}-\alpha\pi^{3n}+\beta\pi^{2n}-\gamma\pi^n+1$) in duas quadraticas reduci poteſt per methodos in analyſi vulgariter notas.

COROLLARIUM V.

Æquationis in genere quadratici diviſores inveniri poſſunt, tranſformando æquationem in alteram, cujus radices ſunt ſummæ ex quibuſque

buſque duabus datæ æquationis radicibus; & e conſtructione æquationis reſultantis (i. e. ex ejus poſſibilibus radicibus) inveniri poſſunt quadratici diviſores quæſiti.

COROLLARIUM VI.

Sit (r) radius, & (a) cujuſlibet arcus ($180 - A$) chorda; & chordæ ſupplementorum arcuum

$$\frac{A}{n},\ \frac{360+A}{n},\ \frac{2\times 360+A}{n},\ \frac{3\times 360+A}{n},\ \text{\&c.}$$

erunt radices æquationis

$$x^n - nr^2x^{n-2} + n\times\frac{n-3}{2}r^4x^{n-4} - \text{\&c.} = a.$$

COROLLARIUM VII.

Dividatur circuli peripheria in (n) æquales partes (AB, BD, DE, EF, FG, &c.) aſſumatur quodcunque punctum (P) in peripheriâ circuli, & ducantur chordæ PA, PB, PD, PE, &c, & erunt hæ chordæ radices æquationis

$$x^n - nr^2x^{n-2} + n\times\frac{n-3}{2}r^4x^{n-4} - \text{\&c.} = a.$$

COROLLARIUM VIII.

Sit (m) impar numerus & minor quam (n), & ſumma (m) poteſtatum ex ſingulis his chordis, erit nihilo æqualis.

COROLLARIUM IX.

Sit (m) minor quam (n), & ſumma ($2m$) poteſtatum ex ſingulis chordis, erit

$$\frac{1\cdot 3\cdot 5\cdot 7\ldots 2m-1}{1\cdot 2\cdot 3\ldots m}\times 2^m\times n\times r^{2m}$$

Conſtant hæc e lemmate primo libri primi, ſed quoniam quadratum chordæ per diametrum i. e. ($2r$) diviſum, æquale eſt verſo ſinui; ſumma (m) poteſtatis ex ſingulis verſubus ſinubus, erit

$$\frac{1\cdot 3\cdot 5\cdot 7\ldots 2m-1}{1\cdot 2\cdot 3\ldots m}\times nr^m.$$

Fig. 26. Aſſumatur punctum (P) quod haud poſitum eſt in circuli peripheriâ, & dividatur eodem modo, quo prior, circumferentia in (n) æquales partes (AB, BD, DE, &c.) a puncto (P) ad hæc puncta diviſionum ducantur.

cantur lineæ PB, PD, PE, &c. producatur linea (CP) jungens centrum & punctum (P), donec peripheriam ſecat in puncto (Q); ducantur lineæ QB, QD, QE, &c; pro lineis PB, PD, PE, &c, ſcribatur reſpective (z); & pro lineis QB, QD, QE, &c. reſpective (x); pro radio, unitas, & pro lineâ (CP), a; tum ſequitur $x^2 = \frac{z^2 + 2a - a^2 - 1}{a}$, qua quantitate pro ſuo valore in æquatione

$$x^n - nr^2x^{n-2} + n \times \frac{n-3}{2} r^4 x^{n-4} - \&c. = 0.$$

ſubſtitutâ, reſultat (ſi n eſt par numerus) æquatio

$$\begin{array}{llll} z^n + \frac{n}{2} \times \overline{2a - a^2 - 1}\, z^{n-2} & + \frac{n}{2} \times \frac{n-2}{4} & \times \overline{2a - a^2 - 1}^{\,2} z^{n-4} & + \&c. = 0 \\ \quad - n \qquad\quad a & - n \times \frac{n-2}{2} & \times a \times \overline{2a - a^2 - 1} & - \&c. \\ & + n \times \frac{n-3}{2} & \times \quad a^2 & + \&c. \end{array}$$

Facile e calculo conſtat ultimum hujuſce æquationis terminum eſſe $a^n \pm 1$, qui æqualis eſt producto ſub ſingulis (n) radicibus.

Si vero (n) eſt impar numerus, tum ſubſtituatur pro (x), $\sqrt{\frac{z^2 + 2a - a^2 - 1}{a}}$, & reſultantis æquationis ultimus terminus etiam erit $a^n \pm 1$.

PROB. XIII.

Datâ æquatione, relationem inter abſciſſam & ejus correſpondentes ordinatas, rectum angulum cum abſciſſa facientes, exprimente Fig. 27.

$$Ay^n + \overline{a + bx}\, y^{n-1} + \overline{c + dx + ex^2}\, y^{n-2} + \&c. = 0.$$

& circa abſciſſæ verticem (P) ducantur lineæ (PA, PB, PC, PD, &c.) æquales angulos inter ſe facientes, invenire æquationem, cujus radices ſunt reſpective lineæ (PA, $P\overset{1}{A}$, PB, $P\overset{1}{B}$, PC, $P\overset{1}{C}$, &c)

Sint l & r duorum arcuum (α & β) coſinus, quorum prior eſt multiplex (m) poſterioris, & radius eſt unitas, tum

$$r = \tfrac{1}{2}\sqrt[m]{l + \sqrt{l^2 - 1}} + \frac{\frac{1}{2}}{\sqrt[m]{l + \sqrt{l^2 - 1}}}$$

&

& r erit reſpective coſinus arcuum $\frac{\alpha}{m}$, $\frac{360-\alpha}{m}$, $\frac{360+\alpha}{m}$, $\frac{720-\alpha}{m}$, & ſic deinceps, i. e. ſi modo peripheria circuli dividatur in $(2m)$ æquales partes, quarum unum punctum diviſionis eſt arcus $\left(\frac{\alpha}{m}\right)$ tum erit (r) coſinus horum arcuum a ſe ipſis æqualiter diſtantium.

Si vero (r) eſt coſinus, tum ſinus erit

$$\sqrt{1-r^2} = \sqrt{\tfrac{1}{2} - \tfrac{1}{4}\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}} - \frac{\tfrac{1}{4}}{\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}}}};$$

ſit z radix æquationis quæſitæ, & abſciſſa (x) erit

$$z \times \tfrac{1}{2}\sqrt[m]{l+\sqrt{l^2-1}} + \frac{\tfrac{1}{2}}{\sqrt[m]{l+\sqrt{l^2-1}}};$$ & ordinata (y)

$$z \times \sqrt{\tfrac{1}{2} - \tfrac{1}{4}\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}} - \frac{\tfrac{1}{4}}{\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}}}};$$ quibus quantitatibus pro ſuis valoribus in data æquatione ſubſtitutis, reſultat æquatio, cujus radix eſt (z).

COROLLARIUM I.

Exterminentur irrationales quantitates, & aſcendere poteſt æquatio reſultans ad $(2nm)$ dimenſiones, ſi vero omnes poteſtates ordinatæ (y) ſunt pares, tum deprimi poteſt æquatio in alteram (nm) dimenſiones habentem.

COROLLARIUM II.

Inventa æquatione, cujus radix eſt (z), invenitur ſumma radicum, contentorum ſub quibuſque duabus, tribus, &c.

Sed ſine ſurdarum quantitatum exterminatione, inveniri poteſt ſumma (n) radicum, contentorum, &c.

Facile eſt has proprietates in ellipſes circa centrum, & quaſdam alias curvas extendere, & contentum ſub ſingulis valoribus invenire.

In prioribus, e datâ æquatione relationem inter abſciſſam & ejus ordinatas exprimente, inventæ fuerunt proprietates; & vice verſâ, e datis proprietatibus inveniri poteſt formula æquationis.

Et

Et ſic de innumeris aliis proprietatibus.

Prius quam huic capiti finem imponam, adjicere viſum eſt tres ſubſequentes propoſitiones.

THEO.

Sit cubica parabola, cujus æquatio relationem inter abſciſſam ($AP=x$) & ejus correſpondentes ordinatas ($PM=y$) exprimens, eſt $p^2x=y^3$. Fig. 28.

Ducatur triangulum EFG, cujus baſis (EF) æquatur ultimæ ordinatæ (PM), altitudo vero abſciſſæ dimidio ($\frac{1}{2}AP$), & angulus $EFG=APM$; biſecentur lineæ AM & EF reſpective in duas æquales partes Ar & rM; $E\alpha$, αF; per punctum (r) ducatur linea parallela axi, ſecans curvam in a, jungantur lineæ Aa & Ma, & ſic per punctum (α) ducatur linea ($\alpha\beta$) parallela lineæ (FG), & triangulum AMa erit æquale trapezio $F\alpha\beta G$.

Biſecentur lineæ Ar, rM, $E\alpha$ reſpective in duas æquales partes As, sr; rt, tM, $E\gamma$, $\gamma\alpha$, & ducantur lineæ sb, tc per (s & t) parallelæ axi; & ſecantes curvam in punctis (b & c), & ſic ducatur linea $\gamma\delta$ parallela lineæ (FG), ducantur etiam lineæ Ab, ba; ac, cM, & ſumma duorum triangulorum ($Aba+acM$) æqualis erit trapezio $\alpha\beta\gamma\delta$, & ſic continuo biſecentur & lineæ (AM) partes, & baſis trianguli (EF,) $E\alpha$, $E\beta$, &c. & figura inſcripta ſemper erit æquale correſpondenti trapezio, & conſequenter triangulum EFG æquale eſt figuræ $AMcabA$ & exinde figura $MPAbacM$ etiam æqualis eſt iſti triangulo.

Conſtat igitur in cubica æque ac in conicâ parabolâ ſummam ex ſucceſſivis triangulis ex continuâ abſciſſæ (AM) partium biſectione ortis, eſſe inter ſe ſemper in ratione numeri quatuor ad unitatem.

Hanc ratiocinandi methodum in alias curvas promovere licet, ſed de hoc in alio loco.

Ejus demonſtratio hoc in loco nimis longa eſt inſerere.

PROB. XIV.

Inſcribere in data ellipſi (EFGH) *parallelogramma, dato parallellogrammati* (ABCD) *ſimile.* Fig. 29.

Inveniantur datæ ellipſeos duæ conjugatæ diametri (EF, GH), quæ angulum interſe æqualem parallellogrammatis angulo (ABC) faciunt.

Sit

Sit C centrum, & in lineâ (EF) capiatur $CP = AB$; per punctum (P) ducatur linea $PQ = BC$, & parallela diametro HG; jungatur linea CQ ſecans curvam in (s), & per (s) ducatur linea (st) parallela diametro HG, & erit parallellogrammatis quæſiti latus; ducantur lineæ sm & tn parallelæ diametro EF, & erunt duo reliqua parallellogrammatis latera.

COROLLARIUM I.

Si data ellipſis duas non habet conjugatas diametros angulum inter ſe dato parallellogrammatis angulo (ABC) æqualem facientes, tum haud admittit ſolutionem problema.

COROLLARIUM II.

Hinc conſtat methodus ducendi duas lineas per centrum datæ conicæ ſectionis, datam inter ſe habentes rationem & datum angulum ſecum facientes.

PROB. XV.

Fig. 30. *Inſcribere in datâ ellipſi* (EFGH) *trapezium, quod erit ſimile dato trapezio* (ABCD).

Ducantur duæ dati trapezii diagonales AC & BD; ſeſe ſecantes in puncto (n), & ſit (nA) major quam Cn; & Bn quam nD.

Secundum problema præcedens ducantur duæ lineæ (LM & RS) per ellipſeos centrum (C) angulum (LCR) inter ſe habentes, angulo(AnB), quem inter ſe faciunt diagonales AC & BD, æqualem, & ita ut $LM^2 : RS^2 :: An \times nC : Bn \times nD$.

Ducantur NP & TV conjugatæ diametrorum LM & RS, ſit etiam $2CP : Pp :: AC : nC$ & $2CT : Tt :: BD : nD$; & ſuper CL & Cp tanquam ſemiconjugatas diametros deſcribatur ellipſis $LphPL$; & ſuper ſemiconjugatas diametros (RC & Ct) deſcribatur ellipſis $(RthSR)$; & interſecent hæ duæ ellipſes ſeſe in puncto (h), per (h) ducantur duæ lineæ (hab & hcd) reſpective parallelæ lineis LM & RS; & erunt quæſiti parallellogrammatis diagonales.

Angulus enim, quem inter ſe faciunt lineæ hab & hcd æqualis eſt angulo (AnB), & diagonalium ſegmenta ah, bh, ch, hd, erunt inter ſe reſpective in ratione linearum An, nC, Bn, nC: & ſic de aliis hujuſmodi propoſitionibus.

LEMMA.

LEMMA.

Omnia parallelogramma circa circulum deſcripta ſunt rhombi, & cujuſcunque parallelogrammi duæ diametri ſeſe interſecant in circuli centro, & rectum angulum inter ſe faciunt.

Et conſequenter duæ diagonales parallelogrammi circa ellipſin deſcripti erunt inter ſe conjugatæ diametri.

PROB. XVI.

Circa datam ellipſim (EFGH) *parallelogrammum deſcribere, quod parallelogrammo* (ABCD) *erit ſimile.* Fig. 31.

Ducantur duæ dati parallelogrammi diametri (AC & BD) ſeſe inter ſecantes in centro N ad datum angulum ANB.

Ellipſeos inveniantur duæ conjugatæ (EF, GH) diametri, quæ faciunt angulum inter ſe æqualem dato angulo ANB, & hæ conjugatæ diametri (EF, GH) juxta præcedens lemma erunt diametri parallellogrammi quæſiti.

Scribantur pro ſemiconjugatis datæ ellipſeos diametris (EF, GH) reſpective t, c; & pro dati parallelogrammi ſemidiametris (AN, NB) reſpective r, s, aſſumatur abſciſſa $CK = \sqrt[2]{\frac{t^4 s^2}{t^2 s^2 + r^2 c^2}}$, per K ducatur ejus ordinata KL; ellipſeos punctum (L) ducatur linea MN tangens, & ſecans diametros EF & GH in M & N, & MN erit latus parallelogrammi quæſiti, quo dato facile etiam deduci poſſunt reliqua latera.

COROLLARIUM.

Si nullæ ſint datæ ellipſeos conjugatæ diametri, quæ angulum inter ſe faciunt æqualem angulo, quem inter ſe habent dati parallellogrammi diametri, tum haud circumſcribi poteſt datæ ſpeciei parallelogrammum.

CAP. II.

PROB. I.

Fig. 32. *Conceſſis curvilinearum quadraturis, rectificare curvoidem, ſeu curvam a puncto in curvâ ſuper rectam rotante generatam.*

Rotetur curva MPL ſuper rectam lineam HG, ſit M punctum, quod deſcribit curvam, cujus rectificatio requiritur, ducatur (PI) tangens ad curvam in puncto P, MI perpendiculum ad tangentem (PI) a dato puncto M; & agatur Lm perpendiculum in axem $= MI$, mr axi parallela $= PI$, & ſecans curvoidem in puncto (n); agatur linea Lr, cui parallela ſit linea (Qn); & erit (Qn) perpendicularis ad curvam in puncto (n).

Ex hypotheſi enim $Lm = MI$, $mr = IP$ & $\angle Lmr = \angle MIP$ & conſequenter $Lr = MP$, & quando tangit axim in puncto (Q) rotans curva (n) erit punctum curvoidem deſcribens, unde linea (Qn) erit perpendicularis ad curvoidem, & conſequenter e ſimilibus triangulis ſequitur curvoidis incrementum $\frac{MP \times \dot{MI}}{PI}$, cujus igitur fluens eſt curvoidis arcus.

EXEMPLUM I.

Fig. 33. Sit curva (MPL) rotans, conici generis parabola, & punctum (M) curvoidem deſcribens, ejus principalis vertex.

Scribantur p & x reſpective pro axis parametro & abſciſsâ a vertice incipiente; tum

$$MI = \frac{\sqrt{p}\,x}{\sqrt{p+4x}},\ MP = \sqrt{px+x^2},\ PI = \frac{\overline{p+2x} \times \sqrt{x}}{\sqrt{p+4x}} \text{ \&c.}$$

$$\frac{MP \times \dot{MI}}{PI} = \frac{\sqrt{p+x} \times \sqrt{p} \times \dot{x}}{p+4x};$$ cujus fluens & ejus conſtructio dantur a viris doctiſſimis Moivræo & Coteſio.

EXEMPLUM II.

Fig. 34. Sit curva (MPI) rotans conici generis ellipſis, & punctum (M) ellipſoidem deſcribens, ejus principalis vertex. Pro ellipſeos majori axi, ejus parametro & abſciſsâ a puncto ellipſoidem deſcribente, incipiente,

ſcri-

ſcribantur reſpective $2a, p, x,$ &

$$MP = \sqrt{\frac{2pax - \overline{p-2a}\,x^2}{2a}},\quad MI = \frac{p^{\frac{1}{2}}ax}{\sqrt{2a \times \overline{2ax - x^2} + p \times \overline{a-x}}}.$$

& $PI = \sqrt{MP^2 - MI^2}$.

Hinc inveniri poteſt ellipſoidis incrementum $\frac{MP \times \dot{MI}}{PI}$. & conſequenter per curvarum quadraturas ejus fluens *i. e.* ellipſoidis arcus.

COR. I.

Iiſdem poſitis, invenire curvoidis aream. Fig. 32.

Linea $mn = mr + rn$ (LQ vel arcui LP), & incrementum areæ curvoidis eſt

$MI \times \text{flux}\, mn = MI \times \overline{\text{flux}\, PI + \text{flux}\, ar\, LP}$; cujus fluens eſt area ipſa.

EXEMPLUM.

Sit curva rotans parabola, & punctum generans, ejus principalis vertex, tum e prædictis conſtat $MI = \frac{\sqrt{p}\,x}{\sqrt{p+4x}}$, & $PI = \frac{\overline{p+2x}\sqrt{x}}{\sqrt{p+4x}}$, Fig. 33.

cujus fluxio eſt $\frac{8x^2 + 3px + \frac{1}{2}p^2}{\overline{p-4x}^{1\frac{1}{2}} \times x^{\frac{1}{2}}}\,\dot{x}$.

& fluxio arcus eſt $\frac{\sqrt{p+4x}\;\dot{x}}{2\sqrt{x}}$; ducantur hæ fluxiones reſpective in

$MI = \frac{\sqrt{p}\,x}{\sqrt{p+4x}}$, & reſultat

$$MI \times \overline{\text{flux}\, PI} = \sqrt{p} \times \frac{8x^3 + 3px^2 + \frac{1}{2}p^2x}{\overline{p+4x}^2 \times x^{\frac{1}{2}}}\,\dot{x},$$

& $MI \times \text{flux}\, arc\,LP = \frac{1}{2}\sqrt{p} \times \sqrt{x} \times \dot{x}$, & conſequenter

$$MI \times \text{flux}\, \overline{PI + \text{flux}\, ArLP} = \overline{\sqrt{p} \times \tfrac{1}{2}\sqrt{x} + \frac{8x^3 + 3px^2 + \frac{1}{2}p^2x}{\overline{p+4x}^2 \times x^{\frac{1}{2}}}} \times \dot{x}$$

cujus fluens innoteſcitur, & eſt area quæſita.

COR. II.

Invenire curvoidis in quocunque puncto (n) curvaturæ radium.

1. Sint

Fig. 32. 1. Sint $\dot{z}, \dot{x}, \dot{y}$ reſpective $\frac{MP \times \dot{MI}}{PI}$, $\dot{MI}$ & $\frac{MI \times \dot{MI}}{PI}$; tum radius curvaturæ, per vulgarium fluxionum principia, in puncto (n), erit $\left(\frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}\right)$.

2. Scribatur pro chorda curvaturæ in puncto P & directione MP, R, tum ſubſequens proportio invenit radium curvaturæ in puncto (n) quæſitum $R - 2Qn\ (MP) : 2Qn :: Qn$: radium quæſitum.

COR. III.

Sit curva generans algebraica curva, & rectificari poteſt generaliter curvoïs, ſi curva generans rectificari poteſt; ſin aliter vero non.

In genere, utrum curva generans eſt algebraica, necne; rectificari generaliter poteſt curvois, ſi ſumma $PI + Ar\ MP$ exprimi poteſt in algebraicis terminis lineæ MI; ſin aliter vero non.

COR. IV.

Datâ curvâ revolvente, invenire, utrum ſunt quædam areæ curvoidis quadrabiles, vel quidam arcus, qui rectificari poſſunt.

Inveniatur (ſi modo poſſibile ſit) area vel arcus generaliter in algebraicis terminis abſciſſæ & ordinatæ, & earum circularium arcuum & Logarithmorum.

Fiant illi termini, ſi modo fieri poteſt, qui involvunt circulares arcus & Logarithmos, nihilo reſpective æquales, & reſultant areæ vel arcus, quæ quadrari, vel qui rectificari poſſunt.

Eodem fere modo inveniri poſſunt areæ vel arcus, qui datam inter ſe habent relationem.

Inveniantur generaliter duæ areæ vel duo arcus, infinitis terminis abſciſſæ & ordinatæ, & earum circularium arcuum & Logarithmorum, & fiant illi termini finiti, circulares arcus & Logarithmi reſpective inter ſe datam habere, ſi modo talem admittunt, relationem, & fit propoſitum.

Si area vel arcus haud inveniri poteſt finitorum terminorum, circularium arcuum vel Logarithmorum ope, tum inveniri poteſt ope infinitarum ſerierum, & e principiis in tractatu de fluxionibus poſtea tradendis erui poteſt problematis ſolutio.

PROB.

PROB. II.

Iiſdem poſitis, rectificare epicurvoidem, i. e. *curvam generatam a puncto in unâ curvâ ſuper alteram tanquam baſim rotante.* Fig. 35.

1. Sit baſis curva (LAS), curva revolvens vero LQP; rectificentur hæ curvæ inter ſe, & invenietur relatio inter lineas LP chordam curvæ revolventis; PQ perpendiculum a puncto (P) in tangentem (LQR) & LQ tangentem, LS ordinatam baſis, SR ejus ſubtangentem, & LR ejus tangentem; pro abſciſsâ (AS) ſcribatur (x), & pro $x - \sqrt{LP^2 - \frac{\overline{PQ \times RS - SL \times LQ}^2}{LR^2}}$ ſcribatur (z), & epicurvoidis arcus fluxio erit $\frac{LP \times LR \times \dot{z}}{PQ \times RS - SL \times LQ}$, cujus fluens eſt epicurvoidis arcus.

2. Sit ($\dot{v}$) incrementum quantitatis $LS + \frac{PQ \times RS - SL \times LQ}{LR}$, & fluxio epicurvoidis arcus erit $\frac{LP \times LR \times \dot{v}}{\sqrt{LP^2 \times LR^2 - \overline{PQ \times RS - LS \times LQ}^2}}$.

3. Datis ad ſingulum punctum (P) radiis curvaturæ & curvæ revolventis, & curvæ ſuper quam tanquam baſim progreditur ea curva. Fig. 36.

Ducantur hi circuli curvaturæ (LM & GH), cujus radii ſint reſpective PD & PC, ſit Q punctum, quod deſcribit epicurvoidem, ducantur etiam lineæ CQ, PS perpendiculum a puncto contactus (P) in lineam (CQ), & DQ, & cum radio DQ deſcribatur circulus ($QmN\tau$), ducantur linea (Pr) per punctum (P) parallela lineæ (DQ), & occurrens lineæ CQ in (r), linea Qv rectum angulum cum linea (PD) faciens, & Qn, Qm, Qo, cotemporanea incrementa epicurvoidis, arcus circuli ($QmN\tau$), & lineæ (CQ).

Facile conſtat lineam Qn eſſe ad rectum angulum ad lineam PQ, & lineam on ad lineam CQ; & exinde duo triangula Qon & PQS eſſe ſimilia, & conſequenter epicurvoidis incrementum $Qn = \frac{PQ \times Qo}{PS}$.

Epicurvoidis incrementum (Qn) etiam erit $\frac{Qm \times QP}{Qr}$; vel etiam (ſi modo

modo TN ſit diameter circuli $TQmN$, incrementum epicurvoidis (Qn) erit $\frac{TN \times CD \times QP \times \text{flux}: TQ}{\sqrt{TN^2 - TQ^2} \times DQ \times CP} = \frac{2\,CD \times QP \times \text{flux}: TQ}{\sqrt{TN^2 - TQ^2} \times CP}$.

Ex his formulis inveniatur generaliter ad ſingulum curvarum punctum epicurvoidis fluxio, & ejus fluens erit arcus epicurvoidis quæſitæ.

EXEMPLUM I.

Invenire epicycloidis longitudinem.

In poſtremâ formulâ incrementum epicycloidis fuit

$Qn = \frac{2\,CD \times QP \times \dot{TQ}}{\sqrt{TN^2 - TQ^2} \times CP}$; ſed in hoc caſu $\sqrt{TN^2 - TQ^2} = NQ = QP$

& conſequenter $Qn = \frac{2\,CD \times \dot{TQ}}{CP}$; ſed CD & CP ſunt in caſu propoſito datæ quantitates, unde incrementum epicycloidis eſt in datâ ratione ad incrementum lineæ QT, horum incrementorum inveniantur & corrigantur integrales; & fit.

EXEMPLUM II.

Fig. 37. Invenire curtatæ vel prolatæ epicycloidis longitudinem.

Scribantur pro lineis, ND, NP, CP, NV reſpective a, b, e, x; unde in formulâ $\frac{2\,CD \times PQ \times \dot{TQ}}{\sqrt{TN^2 - TQ^2} \times CP}$ linea

$$PQ = \sqrt{\overline{b+x}^2\ (PV^2) + 2ax - x^2\ (QV^2)} = \sqrt{b^2 + 2 \times \overline{b+a} \times x}\cdot$$

& quantitates $\sqrt{TN^2 - TQ^2} = QN = \sqrt{2\,ND \times NV} = \sqrt{2ax}$ & $TQ = \sqrt{TV \times TN} = \sqrt{\overline{2a-x} \times 2a}$, & conſequenter ejus fluxio $\dot{TQ} = \sqrt{2a} \times \frac{-\frac{1}{2}\dot{x}}{\sqrt{2a-x}}$, quibus quantitatibus pro ſuis valoribus in prædictâ formulâ ſubſtitutis, reſultat fluxio prolatæ vel curtatæ epicycloidis $\frac{\overline{a+b+e} \times \sqrt{b^2 + 2 \times \overline{b+a}\,x} \times \dot{x}}{\sqrt{2ax - x^2}}$.

COR. I.

Epicurvoidis curvaturæ radius vel inveniri poteſt e duabus primis formulis, e quibus inveniri poſſunt ejus abſciſſæ & ordinatæ fluxiones;

nes; vel ſi S & R ſint reſpective radii curvaturæ & curvæ revolventis & baſis, ad punctum contactus, *i.e.* S & R ſint PD & CD, & $D =$ PQ diſtantiæ puncti curvam generantis a puncto contactus, C chordæ curvaturæ in directione PQ, tum radius curvaturæ epicurvoidis in puncto ($\dot{Q}$) quæſitus, erit $\frac{2 \times \overline{R+S} \times D^2}{2 \times \overline{R+S} \times D - CR}$.

COR. II.

Invenire aream epicurvoidis.

Per problema inveniri poteſt abſciſſa & ejus correſpondens ordinata, & conſequenter productum ex ordinatâ in fluxionem abſciſſæ ductâ, cujus fluens erit area ipſa.

COR. III.

Invenire aream inter curvam, quæ ſit baſis, & epicurvoidem contentam. Fig. 38.

Ducantur duo circuli (LM & GH) ejuſdem curvaturæ ac curvæ revolventis, & baſis, ſuper quam progreditur; cum vero curva revolvens tangat baſim in puncto (P), ſit (a) punctum epicurvoidem deſcribens, rotetur curva a puncto P ad punctum (p), & per (p) ducatur linea ($p\overset{1}{\alpha}$) parallela lineæ (Pa), & linea ph æqualis lineæ (Pa), & eundem angulum cum baſis radio (pC) faciens, quem linea (Pa) facit cum radio (PC) baſeos ad punctum contactus (P), cum vero curva revolvens ponitur ad punctum (p), ſit punctum (l) punctum curvam deſcribens; tum incrementum areæ epicurvoidis inter lineas Pa, Pp, pl, la contentæ, optime diſtingui poteſt in tres partes, una parallelogrammum inter lineas aP & $p\overset{1}{\alpha}$ contentum, ſecunda vero triangulum inter lineas $p\overset{1}{a}$, ph & $\overset{1}{\alpha}h$ contentum, & tertia triangulum inter lineas ph, pl & hl contentum.

Parallelogrammum ($P\overset{1}{\alpha}pa$) æquale eſt baſi Pp ($\dot{A}$) ductæ in perpendiculum an ($\dot{P}$) $= P\dot{A}$.

Triangulum vero $\overset{1}{a}ph$ æquale eſt fractioni $\frac{\overline{aP}^2}{2CP} \times Pp(\dot{A})$, in quâ pro $\dot{a}P$ ſcribe D, & pro CP, R; & fit $\frac{D^2}{2R} \times \dot{A}$.

Duca-

Ducatur enim perpendiculum $(\overset{\prime}{\alpha} r)$ a puncto $(\overset{\prime}{\alpha})$ in lineam (ph); & duo triangula $\overset{\prime}{a}pr$ & CpP ſunt ſimilia, angulus enim PCp æqualis eſt angulo $\overset{\prime}{a}pr$, & conſequenter perpendiculum $\overset{\prime}{\alpha} r = \frac{\overset{\prime}{a}P}{CP} * Pp$, & area trianguli (aph) eſt $\frac{1}{2} ap \times ar = \frac{\overline{aP}^2}{2CP} \times Pp$. Q : E : I.

Triangulum vero phl æquale eſt rectangulo $\frac{1}{2} ph \times hm$ perpendiculum a puncto (h) in lineam (pl); ducatur linea (lq) tangens internum circulum Ll in l, in tangentem lq ducatur perpendiculum (pq) a puncto (p), & erit $hm = \frac{pq \times hl}{pl}$, ſed $hl = \frac{Pp \times El}{PE}$, & conſequenter $\frac{1}{2} ph \times hm = \frac{pq \times El}{PE} \times Pp$, & exinde ſumma horum incrementorum erit $\overline{P + \frac{D^2}{2R} + \frac{pq \times El}{2PE}} \times \dot{A}$, cujus fluens eſt area quæſita.

COR. IV.

Inveniri poteſt, utrum ſint quædam epicurvoidum areæ, &c. quæ exprimi poſſunt in algebraicis terminis datarum linearum, ex principiis, quæ prius data fuerunt.

COROLLARIUM.

Fig. 35. Invenire, utrum epicurvois eſt algebraica curva, necne;

E prædictis conſtat (z) epicurvoidis abſciſſam eſſe

$LV - SA = \sqrt{LP^2 - \frac{\overline{PQ \times SR - LQ \times LS}^2}{LR^2}} - SA$, & ejus ordinatam (y) eſſe $LS + \frac{PQ \times SR - LQ \times LS}{LR}$;

Harum duarum quantitatum (z & y) termini vel ſunt algebraicæ vel fluxionales, ſi ſint algebraicæ tum datur inter eas (z & y) algebraica relatio, & curva eſt algebraica curva, ſi vero ſit fluxionalis, tum poſtea in libro de fluxionibus dabitur ſolutio.

Hinc conſtat, ſi curvæ ſint algebraicæ & rectificari poſſint, tum neceſſario epicurvois erit curva algebraica. Et ſic de reliquis proprietatibus.

PROB.

PROB. III.

Iiſdem poſitis, & datis quibuſlibet numeris curvarum (L M, MK, GF, &c.) *ſuper ſeſe revolventium, rectificare curvam a puncto in extremâ curvâ generatam.* Fig. 39.

Sint velocitates curvarum ſuper ſeſe rotantium reſpective, n, m, r, s, t &c.

Sit prima curva (LM) baſis, ſuper quam progreditur curva (HK), rectificentur hæ curvæ inter ſe, & ducantur circuli, quorum radii ſunt CP & PD reſpective, eandem curvaturam habentes ac datæ curvæ in puncto contactus, (P), ſit (Q) punctum contactus ſecundæ curvæ (HK) & tertiæ (GF), ducatur linea DQ jungens centrum curvaturæ ſecundæ curvæ (HK) & punctum (Q), & radio DQ deſcribatur circulus (QN &c.) jungatur etiam (P) & CQ, & ducatur linea (Pr) parallela lineæ (DQ); & ſint Qo, Qm, Qn cotemporanea incrementa lineæ CQ, arcûs circuli (QN), & epicurvoidis generatæ a puncto (Q) curvæ; tum propter ſimilia triangula Qmn & PrQ, erit

$Qn : Qm :: QP : Pr$; & quoniam incrementa (Qm & Pp) cotemporaneorum arcuum circulorum (QN &c. & HK) ſunt inter ſe in ratione ſuorum radiorum PD & DN, erit

$Qm : Pp :: DN : DP$; quibus rationibus compoſitis, reſultat proportio

$$Qn : Pp :: QP \times DN : Pr \times DP \quad \text{i. e.}$$
$$Qn : Pp :: QP \times CD : CP \times DP \quad \text{i. e.}$$

velocitas puncti Q deſcribentis epicurvoidem eſt ad velocitatem primæ curvæ progredientis ſuper ſecundam in ratione $QP \times CD : CP \times DP$; ſed per hypotheſin velocitas ſecundæ curvæ progredientis ſuper primam eſt ad velocitatem tertiæ curvæ progredientis ſuper ſecundam in ratione $m : n$; & conſequenter velocitas puncti (Q) deſcribentis epicurvoidem eſt ad velocitatem tertiæ curvæ ſuper ſecundam

$$:: m \times QP \times CD : n \times CP \times DP.$$

Sint SQ radius curvaturæ epicurvoidis generatæ a puncto (Q), & QT radius curvaturæ tertiæ curvæ ad punctum contactus (Q); aſſumatur linea $QZ = \frac{m \times QP \times CD \times SQ \times TQ - n \times SQ \times TQ \times CP \times DP}{n \times SQ \times CP \times DP + mQP \times CD \times TQ}$ cum diſtantiis TZ & SZ deſcribantur circuli (Qv & zw), & ſit punctum contactus tertiæ curvæ & quartæ, (k), ducatur linea Zk, & erit

diculum in curvam generatam a puncto (k) in eo puncto, & exinde conſtat curvæ peripheriæ incrementum.

Radius curvaturæ curvæ generatæ a puncto (k) eodem modo inveniri poteſt, quo radius curvaturæ curvæ generatæ a puncto ($\mathcal{Q}$); & exinde inveniri poteſt incrementum peripheriæ curvæ generatæ a puncto in quartâ curvâ rotante ſuper tertiam, & ſic deinceps; & conſequenter inveniri poteſt incrementum peripheriæ curvæ generatæ a puncto in poſtremâ curvâ, & dato ejus incremento, erui poteſt curvæ peripheria.

COR. I.

Methodo ſupra datâ, inveniri poteſt area, &c.

Eadem etiam regula ſæpius repetita invenit radium curvaturæ in pluribus æque ac in duabus curvis ſuper ſeſe revolventibus.

COR. II.

Fig 40. Datis omnibus præter unam curvis, earum velocitatibus, & curvâ generatâ ad ſingula puncta, & exinde deduci poteſt ea curva.

His enim datis, datur diſtantia inter duas curvas, inter quas interponitur quæſita curva, *i. e.* datur chorda quæſitæ curvæ, & incrementum arcus, unde inveniri poteſt curva quæſita.

Ex. Data curvoide, ($\mathcal{Q}\mathcal{Q}$ &c.) i. e. curvâ generatâ a rotatione curvæ ſuper rectam, invenire curvam rotantem.

Datâ curvoide, datur relatio inter abſciſſam & ejus correſpondentem ordinatam, & conſequenter inter $\mathcal{Q}n$ perpendiculum in curvam, & $\mathcal{Q}F$ abſciſſam, & nF; ſit vero (MPL) curva quæſita, tum MP chorda quæſitæ curvæ, MI perpendiculum in tangentem, & PI erunt reſpective æquales lineis $\mathcal{Q}n$, $\mathcal{Q}F$ & nF; & conſequenter datur relatio inter chordam, perpendiculum in tangentem, & tangentem quæſitæ curvæ, unde inveniri poteſt quæſita curva.

Ex his principiis multæ hujuſmodi propoſitiones ſolutionem habere poſſunt.

PROB. IV.

Fig. 41. *Datâ curvâ* (LMN), *invenire alteram* (LRST), *cujus* (*ſi abſciſſa* (LP) *eadem ſit ac datæ curvæ abſciſſa*) *ordinatæ erunt proportionales areis* (LMLC) *datæ curvæ circa datum punctum* C *deſcriptæ.*

Scriba-

Scribatur pro abſciſſa (*LP*) x, pro ordinatâ *PM* (y), & pro *LC*, (a), & fluxio areæ circa datum punctum (*C*) deſcriptæ ſemper erit $y\dot{x}-x\dot{y}+a\dot{y}$; cui fluxioni proportionalis eſſe debet fluxio curvæ quæſitæ ordinatæ ($\dot{v}$), ſi modo ejus abſciſſa eſt (x); inveniatur igitur fluens hujus fluxionis, & erit proportionalis ordinatæ curvæ quæſitæ.

COROLLARIUM.

Sit data curva, circulus vel ellipſis, & erit $y\dot{x}-x\dot{y}+a\dot{y}=\sqrt{2rx-x^2}\,\dot{x}-\frac{\overline{rx-x^2}\times\dot{x}}{\sqrt{2rx-x^2}}+a\dot{y}=\dot{v}=\frac{rx\times\dot{x}}{\sqrt{2rx-x^2}}+a\times\dot{y}$; quæ eſt æquatio ad trochoidem, & conſequenter per trochoidem ſecari poteſt area ellipſeos proportionalis quibuſcunque datis quantitatibus, quam propoſitionem primum invenit Walliſius.

CAP. III.

PROB. I.

Fig. 42. *DATA æquatione relationem inter datæ curvæ* (MmA, &c.) *abſciſſam* (AP) *& ejus correſpondentes ordinatas* (PM) *exprimente, invenire æquationem relationem exprimentem inter abſciſſam* (Ep) *& ejus correſpondentes ordinatas* (pm) *cujuſcunque ſectionis ſolidi e converſione dati ſolidi circa ejus axem generati.*

Sit $r : s : Ep\,(x) : EP$ etiamque $r^2 : t^2 :: \overline{Ep}^2\,(x^2) : \overline{Pp}^2$ ſed $PM^2 - \overline{Pp}^2$ æqualis eſt quadrato lineæ vel ordinatæ (y^2) per punctum (p) ductæ, unde erit datæ curvæ abſciſſa $AP = a\,(AE) + \frac{sx}{r}\,(EP)$ & ejus ordinatæ quadratum $PM^2 = y^2 +- \frac{t^2x^2}{r^2}, \left(\overline{Pp}^2\right)$.

Subſtituantur hæ quantitates pro ſuis valoribus in datâ æquatione & reſultat æquatio relationem inter abſciſſam ($Ep = x$) & ordinatas (y) requiſitæ ſectionis exprimens.

COR. I.

Sectio ſolidi generati a rotatione algebraicæ curvæ (n) dimenſionum circa axem, ſolummodo poteſt eſſe curva ($2n$) dimenſionum.

Si vero nullæ ſunt impares ordinatæ dimenſiones in datâ æquatione, tum ſectio non poteſt eſſe curva plurium quam datæ curvæ dimenſionum.

COR. II.

Sectiones ſolidi generati ex converſione conicæ ſectionis circa ejus axem, erunt conicæ ſectiones, ellipſeos, ellipſis; parabolæ, ellipſis vel parabola, prout axem ſecat, vel ei eſt parallela; hyperbolæ, Ellipſis, parabola vel hyperbola; prout ſecat duas aſymptotos ex eodem latere, parallela eſt uni aſymptoto, vel ſecat oppoſitas hyperbolas.

Et ſic de aliis hyperbolis & parabolis.

COR. III.

Sectiones ſolidi generati a rotatione conicæ hyperbolæ circa ejus axem erunt ſimiles ſectionibus coni, cujus latera ſunt hyperbolæ aſſymptoti, & cujus ſectiones eundem angulum cum axe, quem habent hyperbolæ ſectiones. Sectiones vero ſolidi generati a rotatione pa-

parabolæ circa ejus axem, erunt ſimiles ſectionibus cylindri, quæ eundem angulum cum lateribus cylindri, ac ſectiones paraboloidis cum ejus axe faciunt.

Generaliter vero, ſint duæ conicæ ſectiones (*ABCD* & *EFGH*), quarum duæ æquationes relationem inter utriuſque curvæ abſciſſas (z & w) ad axes & earum correſpondentes ordinatas, ſunt reſpective, Fig. 43.

$$v^2 = \frac{c^2}{t^2}\overline{t^2 \pm z^2} \text{ \& } u^2 = \frac{C^2}{T^2}\overline{T^2 \pm w^2};$$

rotentur hæ curvæ circa ſuos axes, & ſolidi generati a rotatione curvæ (*ABCD*) ſit ſectio ($\alpha\beta\gamma\delta$) ſimilis ſectioni ($\epsilon\zeta\eta\theta$) ſolidi (*EFGH.*)

Prior vero ſectio cum axe ejus ſolidi faciat angulum, cujus ſinus eſt (S); poſterior vero cum ejus ſolidi axe angulum, cujus ſinus eſt (s), tum erit

$$S^2 = \frac{\overline{1 \pm \frac{C^2}{T^2}} \times s^2 \pm \frac{c^2}{t^2} \pm \frac{C^2}{T^2}}{1 \pm \frac{c^2}{t^2}}$$

ſubſtituantur enim pro (z & v^2) per problema in priori æquatione $\sqrt{1 - S^2}\, x$ (ſectionis abſciſſam) $+ a$, & y^2 ejus ordinatæ quadratum $+ S^2 \times x^2$; in poſteriori æquatione pro (w & u^2) reſpective $\sqrt{1 - s^2}\, \zeta$ (ſectionis abſciſſam) $+ e$, & ξ^2 (ejus ordinatæ quadratum) $+ s^2\zeta^2$, & reſultant duæ æquationes

$$y^2 + S^2x^2 = c^2 \pm \frac{c^2 \times \overline{1 - S^2}}{t^2} x^2 + \text{\&c.}$$

$$\xi^2 + s^2\zeta^2 = C^2 \pm \frac{C^2 \times \overline{1 - s^2}}{T^2} \zeta^2 + \text{\&c.}$$

ſint hæ duæ ſectiones inter ſe ſimiles, & coefficiens quadrati ordinatæ ad coefficientem quadrati ejus abſciſſæ in utraque æquatione eandem habet rationem, & conſequenter

$$S^2 \mp \frac{c^2 \times \overline{1 - S^2}}{t^2} = s^2 \mp \frac{C^2 \times \overline{1 - s^2}}{T^2},$$

reducatur hæc æquatio, & fit

$$S^2 = \frac{\overline{1 \pm \frac{C^2}{T^2}}\, s^2 \pm \frac{c^2}{t^2} \mp \frac{C^2}{T^2}.}{1 \pm \frac{c^2}{t^2}}$$

CO-

COR. IV.

Fig. 44. Conica ſectio *ABG* circa ejus axem (*BC*) rotetur, & ſit ſectio ſolidi generati (*DEF*); ducatur plano (*DEF*) parallelum planum (*HIK*), tangens ſolidum in puncto (*I*); biſecetur axis ſectionis (*DF*) in *L*, & jungatur punctum contactus (*I*) & *L*, & hæc linea producta tranſit per omnia centra ſectionum parallelarum ſectioni (*DEF*).

Similiter ſit (*Q*) focus conicæ ſectionis, cujus rotatione generatur ſolidum, ſit *DEF* hujus ſolidi ſectio, cui ſit planum (*HIK*) tangens ſolidum parallelum; ſit linea (*IN*) diameter a puncto contactus (*I*) per centrum ſolidi tranſeuns; ſuper lineam tanquam axem per punctum (*Q*) deſcribatur conica ſectio, & hæc conica ſectio per focos omnium ſectionum parallelarum ſectioni (*DEF*) tranſibit.

Et ſic inveniri poſſunt centra, foci &c. ſectionum a planis circa data puncta rotantibus factarum, & conſequenter locos centrorum, focorum, &c.

PROB. II.

Fig. 45. *Datis duabus æquationibus, una, quæ exprimit relationem inter primam abſciſſam* (AP) *& ejus ordinatas in eodem plano inſiſtentes* (PM), *altera vero, quæ exprimit relationem inter ordinatam* (PM), *& abſciſſam* (AP), *partes ordinatæ* PM (Pp) *tanquam ſecundam abſciſſam, & ejus ordinatas* (pm) *per punctum abſciſſæ* (p) *tranſeuntes, invenire æquationem relationem inter abſciſſam & ejus correſpondentes ordinatas cujuſcunque ſectionis* (LMN) *hujus ſolidi exprimentem.*

Scribantur pro ſinubus angulorum, quos ſectio facit cum axe *AP* (*pLP*), cum ejus ordinatis *PM* (*LpP*), & ordinatæ faciunt cum ſuo axe (*MPL*), reſpective, *t*, *r*, *s*; & ſubſtituatur pro diſtantiâ inter punctum, in quo ſectio ſecat axem, & abſciſſæ axis verticem (*AL*) *a*; & pro abſciſſa (*Lp*) & ejus ordinatâ (*pm*) ſectionis quæſitæ reſpective *x* & *y*; tum erit

Sin $\angle pPL\,(s) = \text{ſin}\; PpL\,(r) :: Lp\,(x) : LP$, & exinde $LP = \frac{rx}{s}$;

& axis abſciſſa $(AP = \frac{rx}{s} + a)$; etiamque

Sin $\angle pPL\,(s) : \text{ſin}\,\angle\, pLP\,(t) :: Lp\,(x) : Pp$; unde ſecunda abſciſſa $(pP) = \frac{tx}{s}$.

Sub-

Subſtituantur hæ quantitates $\left(\frac{rx}{s}+a \,\&\, \frac{tx}{s}\right)$ pro ſuis valoribus (AP & Pp) in datâ æquatione, & reſultant duæ æquationes tres incognitas quantitates (x, y & PM) habentes, reducantur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (PM), & reſultat æquatio relationem inter incognitas quantitates (x & y) abſciſſam & correſpondentes ordinatas ſectionis quæſitæ.

COR. I.

Nonnunquam prius neceſſe eſt mutare inclinationem ordinatarum (pm) ad ſecundam abſciſſam (ap), & deinde prædictâ methodo progrediendum eſt.

COR. II.

Hujus ſolidi ſectio non poteſt habere plures dimenſiones, quam habet productum ſub dimenſionibus duarum datarum æquationum vel curvarum in ſeſe ductis.

COR. III.

Fiat curva MMM, &c. rectalinea, & ſectiones erunt curvæ, quarum dimenſiones ſunt æquales datæ curvæ dimenſionibus, ergo curva ſemper projiciat curvas dimenſionum ſibi ipſi æqualium.

COR. IV.

Sit curva MM, cujus æquatio, relationem inter abſciſſam (Pp) & ejus correſpondentes ordinatas (pm) per punctum (p) tranſeuntes exprimens, datur; & A punctum lucidum, invenire projectionem datæ curvæ in datum planum LpM. Fig. 46.

1mo. Sin $\angle LPp\,(s)$: ſin. $PpL\,(r) :: pL\,(x) : PL$; ergo $PL=\frac{r\times x}{s}$ & $PL+LA=AP=\frac{rx}{s}+a$; & eodem modo ſin $\angle LPp\,(s)$: ſin $\angle PLp\,(t) :: pL\,(x) : Pp$, & conſequenter abſciſſa $Pp=\frac{tx}{s}$; ſit vero $p\times AP=PM$, & exinde $PM=\frac{prx}{s}+pa$; ſubſtituantur hæ quantitates

tates $\left(\frac{rx}{s}+a, \frac{tx}{s}, \frac{prx}{s}+pa\right)$ pro ſuis valoribus AP, Pp & PM, & reſultat æquatio relationem inter abſciſſam ($Lp=x$) & correſpondentes ordinatas (pm) quæſitæ ſectionis exprimens.

Parallelæ ſectiones hujus generis ſolidi erunt inter ſe ſimiles.

In nonnullis caſibus primum mutare ordinatarum inclinationem neceſſe eſt.

COR. V.

Hinc conſtat curvas, quarum æquationes relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimentes, ſunt

$$y^{2n}+\overline{c+dx+ex^2}\,y^{2n-1}+\&c.=0$$

projicere poſſet omnes curvas ($2n+1$) dimenſionum.

Eodem modo invenire licet curvas, quas projicere poſſunt quæcunque curvæ, & vice verſâ.

Ex æquatione etiam inveniri poteſt, annon ſolidum dividi poteſt in duas vel plures æquales partes.

Ex iiſdem principiis, quibus uſus fui in hujus libri capite primo inveniri poſſunt æquationes inter quaſcunque alias lineas, ad hujuſmodi ſolidi abſciſſas & ordinatas quamcunque aſſignabilem relationem habentes.

PROB. III.

Fig. 47. *Dato puncto lucido* (Q) *& duabus æquationibus, e quibus una exprimit relationem inter abſciſſam* (AP) *& ejus correſpondentes in eodem plano ordinatas* (PM), *altera vero exprimit relationem inter ordinatam* (PM), *abſciſſam* (AP), *& partes* (Pp) *ordinatæ* PM *tanquam ſecundam abſciſſam, & ejus ordinatas* (pm) *per punctum ſecundæ abſciſſæ* (p) *tranſeuntes, invenire ejus projectionem in datum planum.*

Datum planum tranſeat per axem AP in directione ordinatarum ad ſecundum axem, & ita transformetur ordinatarum (PM) inclinatio ad primum axem, vel quod idem eſt ſecundæ abſciſſæ (Pp) planum, ut fiat idem ac planum linearum jungentium punctum lucidum (Q) & puncta axis (AP).

Ducatur QH parallela ordinatis (PM), occurrens axi (AP) in H, ſit (L) quodlibet axis (AP) punctum ducatur QL, ducantur ad partem

tes lineæ $\mathcal{Q}L$ correſpondentes ordinatæ (pm, &c.) ducatur linea $\mathcal{Q}m$ tangens ſolidum in (m) & ſecans datum planum per axem tranſeuns in o, per punctum (p) ducatur linea (pP) parallela ordinatis (PM) vel lineæ $\mathcal{Q}H$, & a puncto ($\mathcal{Q}$) perpendiculum in axem ($\mathcal{Q}h$).

Scribantur pro lineis $\mathcal{Q}H, AH, \mathcal{Q}h, hH, HL, Lo, Lp, pm, Pp, PH$, reſpective, $a, b, c, d, x, y, X, v, z, w$; in duabus datis æquationibus ſubſtituantur pro ($HP = w$ & $Pp = z$) earum reſpectivi valores

$$\frac{\sqrt{c^2 + \overline{x-d}^2} \times \overline{x+b} - xX}{\sqrt{c^2 + \overline{x-d}^2}}, \ \& \ \frac{aX}{\sqrt{c^2 + \overline{x-d}^2}};$$ & reſultant duæ æquationes, quibus in unam reductis ita ut exterminetur incognita quantitas (PM), reſultat æquatio relationem inter abſciſſam Lp (X) & ejus ordinatas pm (v) exprimens; inveniatur ſubtangens hujus curvæ, quæ fiat æqualis quantitati $\sqrt{c^2 + \overline{x-d}^2} - X$; & in æquatione reſultanti pro $pm = v$ ſubſtituatur ejus valor $\frac{y \times \overline{\sqrt{c^2 + \overline{x-d}^2} - X}}{\sqrt{c^2 + \overline{x-d}^2}}$, & reſultant duæ æquationes tres incognitas quantitates (x, y, X) habentes, quibus in unam reductis, ita ut exterminetur incognita quantitas (X), reſultat æquatio relationem inter abſciſſam (x) & ejus correſpondentes ordinatas ſolidi projectionis in datum planum exprimens.

EXEMPLUM I.

Sit ſolidum cylindricum, cujus baſis eſt quæcunque curva, & ejus laterum projectio in datum planum erit rectæ lineæ.

EXEMPLUM II.

Sit ſolidum conus, cujus baſis eſt quæcunque curva, & ejus laterum projectio in datum planum erit rectæ lineæ.

Facile conſtant hæc etiam ex vulgari trigonometriâ & fluxionibus.

COROLLARIUM

Si planum non tranſit per axem, & haud eandem inclinationem in axem cum ejus ordinatis habet, e præcedenti problemate & ejus corollariis facile deduci poteſt ejus projectio.

PROB. IV.

Fig. 48. *Datis, puncto lucido (Q) & duabus æquationibus ut in duabus præcedentibus propoſitionibus, & tertiâ relationem inter abſciſſam ($\alpha\pi$) & ejus correſpondentes ordinatas ($\pi\mu$) datæ curvæ exprimente, invenire projectionem datæ curvæ in datum ſolidum.*

Transformetur abſciſſa curvæ vel axis ſolidi, ita ut fiat projectio abſciſſæ curvæ in dati ſolidi axem, & ita transformetur ſecunda abſciſſa, ut ſit in eodem plano cum curvæ & ſolidi axe, & ſecundæ abſciſſæ ordinatæ (pm), ita ut fiant parallelæ plano datæ curvæ; vel ordinatæ datæ curvæ ($\pi\mu$), ut fiant parallelæ ordinatis (pm) ſecundæ abſciſſæ ſolidi.

Ducatur linea QH parallela ordinatis (PM), & occurrens abſciſſæ curvæ in (r) & axi ſolidi in (H).

Ducatur per punctum curvæ (μ) linea ($Q\mu$) occurrens ſolido in puncto (m), & per ejus correſpondens punctum abſciſſæ (π) linea Qp.

Scribantur pro lineis $Qr, QH, \alpha r, HA, AP, Pp, pm, \alpha\pi, \pi\mu$, reſpective $a, b, c, d, x, z, v, X, Y$; tum propter duo ſimilia triangula $Q\pi\mu$ & Qpm, erit $Qr : QH - Pp :: \pi\mu : pm$, i. e. $a : b - z :: Y : v$, & ex iiſdem principiis etiam ſequitur ſubſequens proportio

$$Qr : QH - Pp :: \alpha\pi - \alpha r : AP - AH, \text{ i. e.}$$
$$a : b - z :: X - c : x - d;$$ e primâ proportione ſequitur $\frac{av}{b-z} = Y$, e ſecundâ vero

$X = \frac{ax - ad + bc - zc}{b - z}$; quibus quantitatibus pro ſuis valoribus in æquatione relationem inter abſciſſam curvæ (X) & ejus ordinatas (Y) exprimente, reſultant tres æquationes, quatuor incognitas quantitates (x, z, v, PM) habentes, quibus in unam reductis, ita ut exterminetur incognita quantitas (PM) & una e reliquis (x, z, v), reſultat æquatio, relationem inter duas reliquas exprimens, quæſita.

COR. I.

Dimenſiones æquationis reſultantis non plures eſſe poſſunt, quam contentum ſub dimenſionibus trium datarum æquationum, i. e. ſint, m, n & r datarum æquationum dimenſiones, & reſultans æquatio non plures quam $n \times m \times r$ dimenſiones habere poteſt.

COR.

COR. II.

Si vero axis ſolidi non ſit projectio abſciſſæ curvæ ſit, & ordinatæ ſecundæ abſciſſæ ſolidi ordinatis haud ſint parallelæ, & ſecunda abſciſſa ſolidi haud ſit in eodem plano cum axe ſolidi & abſciſsâ curvæ, tum ex hoc problemate & capite primo petenda eſt ſubſtitutio.

COR. III.

Solidi illuminati contentum inveniri poteſt e principiis fluxionum, inveſtigando aream curvæ contentam inter lineas (*mpm*) & arcum curvæ *mRSm*; & ducendo hanc aream in fluxionem abſciſſæ (*AP*), & erit productum fluxio ſolidi fruſtri quæſiti.

COR. IV.

Arcûs curvæ umbram terminantis longitudo inveniri poteſt, inveſtigando fluentem fluxionis $\sqrt{\dot{z}^2+\dot{w}^2+\dot{v}^2}$, in quâ $\dot{z}, \dot{w}, \dot{v}$ reſpective denotant fluxiones abſciſſæ, ordinatæ & diſtantiæ a plano, inter ſe rectum angulum facientes.

Ex iiſdem etiam principiis inveniri poſſunt generaliter curvarum ſuperficies.

Hic inſerere licet curvoidalem trigonometriam, quæ in ſuo calculo multum habet laborem, ſed ejus principia ſatis perſpicua ſunt; etiamque multas alias propoſitiones de ſolidis verſantes: ſed ſatis quoad noſtrum propoſitum de his dictum eſt.

CAP. IV.

DEF. I.

LINEÆ fluxionales, ſecundum ordines fluxionum æquationis, quâ relatio inter abſciſſas & ordinatas definitur, diſtinguuntur in ordines; ea linea eſt primi fluxionalis ordinis, in cujus æquatione relationem inter ejus abſciſſas & ordinatas exprimente, continentur ſolummodo fluxiones primi ordinis, & ſic deinceps.

DEF. II.

Fluxionales lineæ ejuſdem ordinis, ſecundum dimenſiones æquationis, quâ exprimitur relatio inter abſciſſas (x) & ordinatas (y), ſi modo pro fluxione abſciſſæ ($\dot{x}$), vel ordinatæ ($\dot{y}$), ſcribatur x & y reſpective, & pro ($\ddot{x}$) vel ($\ddot{y}$), ſcribatur x^2 vel y^2; & ſic deinceps, diſtingui poſſunt in dimenſiones.

PROB. I.

Fig. 50. *Datâ fluxionali æquatione relationem inter abſciſſam* (AP, x) *& ejus ordinatas* (PM, y) *exprimente, eam in alteram transformare, relationem inter abſciſſam* (ap, z) *& ejus correſpondentes ordinatas* (pm, v) *exprimentem.*

Subſtituantur pro abſciſsâ (x), & ejus ordinatis (y), eædem quantitates, quæ in problemate primo ſubſtitutæ fuerunt, & pro abſciſſæ (x) & ordinatarum (y) fluxionibus ſubſtituantur prædictarum quantitatum correſpondentes fluxiones; & transformatur æquatio in alteram, relationem inter abſciſſam (ap) & ejus ordinatas (pm) exprimentem.

THEO.

Harum curvarum multæ ſunt proprietates algebraicarum curvarum proprietatibus analogæ, quæ per methodum convergentium ſerierum, erui & demonſtrari poſſunt.

EXEMPLUM I.

Sit fluxionalis curva (n) dimenſionum, & ducantur duæ rectæ, ſibi ipſis parallelæ, & curvam ſecantes in (n) punctis; linea, quæ ita ſecat

cat has parallelas, ut ſumma partium ex uno ſecantis latere, æqualis ſit ſummæ partium ex altero ſecantis latere, eodem modo ſecabit omnes alias his parallelas.

COR. I.

Rejiciantur omnes datæ fluxionalis æquationis termini, iis exceptis, in quibus inveniuntur n & $n-1$ dimenſiones; æquationis reſultantis inveniatur fluens, quæ ſit algebraica curva, & diameter hujus algebraicæ curvæ erit etiam fluxionalis curvæ diameter, & vice verſa.

COR. II.

Sint (n) aſymptoti fluxionalis curvæ, cujus numerus dimenſionum eſt (n); & agatur recta linea, curvam in (n) punctis ſecans; tum iſtius lineæ ſumma partium, inter crura curvæ & eorum reſpectivas aſymptotos interceptarum, ex uno diametri latere, æqualis eſt ſummæ partium inter curvæ crura, & eorum reſpectivas aſymptotos interceptarum, ex altero diametri latere.

EXEMPLUM II.

Datâ fluxionali curvâ, in quâ nulli inveniuntur termini $(n-1)$ dimenſionum, & generale centrum habet curva.

Eodem modo progredi licet in curvas, omnes ſuas diametros parallelas habentes.

EXEMPLUM III.

Sint omnes ordinatæ dimenſiones in datâ fluxionali æquatione, pares; & correſpondentes ordinatæ valores, & affirmativi & negativi, erunt inter ſe æquales.

EXEMPLUM IV.

Facile e methodo convergentium ſerierum erui poteſt, quot ſunt crura parabolica, hyperbolicave; pendent enim e terminis datæ æquationis, quorum dimenſiones ſunt maximæ, & inveniri poſſunt eâdem methodo, quâ parabolica vel hyperbolica algebraicæ curvæ crura.

PROB. II.

Datâ fluxionali æquatione relationem inter abſciſſam (x), *&: ejus correſpondentes ordinatas* (y) *exprimente, invenire abſciſſas & ordinatas, quando duo valores ordinatæ fiunt inter ſe æquales.*

In-

Inveniatur datæ æquationis fluxio, & fiat ordinatæ fluxio, vel nihilo, vel infinito, æqualis, & reſultant duæ fluxionales æquationes, duas incognitas quantitates (x & y) & earum fluxiones habentes, quæ reduci poſſunt in unam algebraicam æquationem, unam ſolummodo incognitam quantitatem (x vel y) habentem, ut poſtea docebitur in noſtro de fluxionibus tractatu.

COR. I.

Hinc inveniri poſſunt ovales, puncta conjugata, &c. ex iiſdem principiis, quæ in capite primo de inveniendis ovalibus, punctis conjugatis, &c. in algebraicis curvis, tradita fuerunt.

COR. II.

Hinc erui poteſt fluxionalium curvarum, quarum dimenſiones dantur, enumeratio.

Datur enim methodus inveniendi curvæ parabolica, hyperbolicave crura, & eorum aſymptotos, etiamque e præcedentibus principiis inveniri poſſunt puncta, in quibus aſymptotos ſecat curva.

Datur etiam methodus inveniendi ovales, puncta conjugata, &c. & conſequenter erui poteſt fluxionalium curvarum, quarum dimenſiones dantur, enumeratio.

PROB. III.

Curvæ primi fluxionalis ordinis conſtrui poſſunt, curvarum generatarum ex rotatione algebraicarum ſuper rectas, i. e. curvoidum & algebraicarum ope.

Curvæ ſecundi fluxionalis ordinis conſtrui poſſunt, curvarum ex rotatione algebraicarum curvarum ſuper algebraicas generatarum, i. e. epicurvoidum, curvoidum & algebraicarum curvarm ope, & ſic deinceps.

Hic adjicere liceret proprietates fluxionalium curvarum, fere omnibus, vel algebraicis vel fluxionalibus algebraicarum curvarum proprietatibus, analogas.

Sed plura adjiciam de his fluxionalibus curvis in tractatu de fluxionibus: progredior autem ad tractandas curvarum proprietates, quarum relatio inter abſciſſam & ejus correſpondentes ordinatas haud deſignatur per æquationem, vel fluxionalem, vel algebraicam; quæ-

pro-

proprietates in unâ curvâ pendent ex ejus concavitate vel convexitate, vel quibuſdam datis vel cognitis proprietatibus; in pluribus autem curvis, e datis æquationibus relationem inter quaſdam lineas in unâ, & quaſdam in aliis curvis, definientibus: ſed priuſquam harum curvarum nonnullas proprietates inveſtigo, neceſſe eſt ſubſequentia lemmata præmittere.

LEMMA I.

Dividatur ſingulum trianguli (*EFG*) latus in duas æquales partes, *EA*=*AG*, *GB*=*BF* & *FC*=*CE*; jungantur puncta biſectionum *A*, *B*, *C*; & erit circumſcriptum triangulum ad inſcriptum in ratione numeri quaternarii ad unitatem. Fig. 51.

LEMMA II.

Dividatur ſingulum latus trapezii *EFGH* in duas æquales partes *EA*, *AF*; *FB*, *BG*; *GC*, *CH*; *HD*, *DE*; jungantur puncta biſectionum *ABCDA*, & circumſcriptum trapezium erit ad inſcriptum in ratione numeri binarii ad unitatem. Fig 52.

DEMONSTRATIO.

Ducantur diagonales *EG*, *FH* & ex hypotheſi *AE*=*AF*, & *FB*=*BG*, & conſequenter triangulum (*AFB*) erit quarta pars trianguli (*FEG*); & ſic triangulum (*DHC*) erit quarta pars trianguli (*HEG*), & ſumma duorum triangulorum (*FAB*+*DHC*) erit quarta pars dati trapezii (*EFGH*); & ſic ſumma duorum triangulorum (*AED*+*BGC*) erit quarta pars dati trapezii; ergo ſumma quatuor triangulorum (*AFB*+*BGC*+*CHD*+*DEA*), & exinde trapezium (*ABCD*) erit ad datum trapezium (*EFGH*) in ratione unitatis ad numerum binarium.

COROLLARIUM.

Lineæ (*AB* & *DC*) ab *Euclide* demonſtrantur eſſe parallelæ diagonali (*EG*), & conſequenter parallelæ ſunt inter ſe; & ſimiliter demonſtrantur lineæ (*BC* & *AD*) inter ſe parallelæ: ergo inſcriptum trapezium (*ABCD*) eſt parallelogrammum.

In omnibus polygonis hâc methodo inſcriptis, area polygoni inſcripti erit ac aream dati polygoni circumſcripti, in majore ratione quam unitatis ad numerum binarium.

LEMMA

LMMA III.

Fig. 53. Dato quocunque trapezio (*ABCD*), cujus diagonales ſunt lineæ (*AC* & *BD*), per puncta (*A*, *B*, *C*, *D*) ducantur lineæ diagonalibus (*AC* & *BD*) parallelæ, & erit figura exinde formata (*EFGH*) parallelogrammum.

Lineæ enim (*FG* & *EH*) ducuntur parallelæ lineæ (*AC*), & igitur ſunt parallelæ inter ſe, & eodem modo probatur lineas (*EF* & *GH*) parallelas eſſe inter ſe, & conſequenter trapezium (*EFGH*) eſſe parallelogrammum.

COROLLARIUM.

Circumſcriptum parallelogrammum (*EFGA*), duplum eſt dati trapezii *ABCD*).

Parallelogrammum enim (*AG*) duplum eſt trianguli (*ABC*), & parallelogrammum (*AH*) duplum eſt trianguli (*DAC*), ergo totum parallelogrammum (*EG*) duplum eſt trapezii (*ABCD*).

Conſtat polygonum hâc methodo circumſcriptum, non habere ad datum polygonum inſcriptum, majorem rationem, quam numeri binarii ad unitatem.

LEMMA IV.

Fig. 54. Sit polygonum (*EFGHI*, &c.) & ducantur lineæ *AB*, *BC*; *CD*, *DE*; &c. ita ut angulus *ABE* æqualis ſit angulo *CBF*; angulus *BCF* ſit æqualis angulo *DCG*, $\angle CDG = \angle \overset{\prime}{D}DH$, &c.

1. Sit numerus polygoni laterum impar, & ſcribantur pro angulis dati polygoni *EFG*, *FGH*, *GHI*, &c. reſpective a, b, c, d, e, &c. tum erit angulus

$$CBF = \frac{180 - a + b - c + d - e + \text{\&c.}}{2}\text{; \& ſic de}$$

reliquis.

DEMONSTRATIO.

Scribantur pro angulis *CBF*, *BCF*, *CDG*, $D\overset{\prime}{D}H$, $\overset{\prime}{D}\overset{\prime\prime}{D}I$, &c. reſpective x, y, z, v, w, &c. & reſultant æquationes

$$a + x + y = 180^\circ$$
$$b + z + y = 180$$
$$c + v + z = 180$$
$$d + v + w = 180$$
$$e + x + w = 180$$

&c. &c. ita reducantur hæ æquationes in unam

unam, ut exterminentur omnes incognitæ quantitates præter unam, quæ ſit (x), & fit

$$x = \frac{180 - a + b - c + d - e + \&c.}{2}$$

Si quicunque angulus inventus, ſit negativus, tum tale polygonum inſcribi nequit.

2. Sit numerus polygoni laterum par, & hujuſce generis polygonum inſcribi nequit, ni

$$a - b + c - d + e - \&c. = 0.$$

COROLLARIUM I.

Scribantur pro lineis EF, FG, GH, HI, IK, &c. reſpectīve P, Q, R, S, T, &c. pro linea EA ſcribatur (x), ſit $EA : EB :: 1 : o$; $BF : FC :: 1 : n$; $CG : GD :: 1 : m$; $DH : H\overset{1}{D} :: 1 : l$; &c. tum

$$EB = ox$$
$$FC = n \times \overline{EF - EB} = nP - nox$$
$$GD = m \times \overline{FG - FC} = mQ - mnP + mnox$$
$$H\overset{1}{D} = l \times \overline{GH - GD} = lR - lmQ + lmnP - lmnox$$

& ſic deinceps.

Et exinde e ſimplici æquatione ſequitur valor incognitæ quantitatis (x), quo ſubſtituto, reſultant reliquarum incognitarum quantitatum valores.

COROLLARIUM II.

Sit polygonum (n) laterum triangulum (EFG), & ſumma laterum ($EF + FG + GH$) non habet ad ſummam linearum ($AB + AC + BC$) majorem rationem quam 2 : 1. Fig. 50.

Sit polygonum trapezium, & ſumma laterum ($EF + FG + GH + HE$) Fig. 51. non habet ad ſummam linearum ($AB + BC + CD + DA$) majorem rationem quam $\sqrt{2} : 1$, & conſequenter ſumma laterum polygonorum plura latera habentium minorem habent ad ſummam prædictarum linearum rationem, quam $\sqrt{2} : 1$.

LEMMA V.

Sit polygonum (n) laterum ($EFGH$, &c.) & ita biſecetur ſingulum latus in punctis (A, B, C, &c.) ut ejus ſegmenta ſint inter ſe in Fig. 54.

reciprocâ ratione ſummæ ſecantis & tangentis complementorum adjacentium angulorum, i. e. ſit

$EB : BF : Sec + Tan$ Com : $\angle EFG$: $Sec + Tan$ Com : $\angle FEH$

& $FC : CG : \overline{Sec + Tan}$ Com : $\angle FGH$: $\overline{Sec + Tan}$ Com : $\angle GFE$

& ſic deinceps.

Jungantur puncta contactus A & B, B & C, C & D, &c. & circumſcripti trapezii vel polygoni ſumma laterum $EF+FG+GH+$&c. non majorem habet rationem ad inſcripti trapezii vel polygoni ſummam laterum $AB+BC+CD+$ &c. quam $\sqrt{2} : 1$.

Sit datum polygonum triangulum, & circumſcripti trianguli laterum ſumma, non poteſt habere ad inſcripti trianguli laterum ſummam, majorem rationem, quam numerus binarius ad unitatem.

Et ſic de quam plurimis hujuſmodi propoſitionibus.

PROB. IV.

Invenire curvarum proprietates, quarum nulla datur æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas deſignans.

Proprietates ejuſdem curvæ, vel ad unum ſolummodo curvæ punctum, vel ad plura, vel ad omne punctum referuntur.

1. Si ad unum ſolummodo punctum referuntur lineæ, inter quas requiruntur æquationes relationem exprimentes.

Hæ proprietates e vulgaris analyſeos & fluxionum principiis petendæ ſunt.

EXEMPLUM.

Ita transformare æquationem fluxionalem, in quâ ſolummodo continentur duæ variabiles quantitates (x & y), quarum (x) fluit uniformiter, ut fluat uniformiter (y).

Fig. 55. Sit quæcunque curva (MRS), cujus abſciſſa ($x=AP$) fluit uniformiter, ſit ejus fluxio, (Pp), & ſit fluxio ordinatæ ($y=PM$), Mh; ſecunda vero fluxio $2mf$; nunc fluat uniformiter ordinata (PM), ſit ejus fluxio (Mh), tum prima abſciſſæ (AP) fluxio erit he, & ſecunda ejus fluxio $2me$: duo triangula Mhe, fme ſunt ſimilia; ergo eh ($\dot{x}$) : hM ($\dot{y}$) : : $2em$ ($\ddot{x}$) ſecunda abſciſſæ fluxio, cum fluat uniformiter (y) : $2mf$ ($\ddot{y}$) ſecundam ordinatæ fluxionem, cum fluat uniformiter

miter (x); & conſequenter $\ddot{y} = \frac{\dot{y}\ddot{x}}{\dot{x}}$, quo valore pro $(\ddot{y})$ in datâ fluxionali æquatione ſubſtituto, & ejus fluxione pro $\dot{\ddot{y}}$, & ſic deinceps; & ita transformabitur æquatio, ut fluat uniformiter (y).

Hinc ſequitur omnis hujuſce generis transformatio.

2. Si vero ad plura puncta referuntur lineæ, inter quas requiruntur æquationes relationem exprimentes.

Data relatio vel exprimitur per algebraicas vel fluxionales æquationes, ſi vero per algebraicas, tum ex vulgaris analyſeos & geometriæ principiis petenda eſt ſolutio.

Si vero per fluxionales æquationes exprimitur data relatio, tum inveniendæ ſunt lineæ, quæ inter ſe eaſdem habent rationes, quas habent inter ſe fluxiones, quibus pro fluxionibus reſpective ſubſtitutis, transformantur fluxionales æquationes in algebraicas.

Sequuntur hujus caſus quædam exempla, de inveniendis, quibuſdam areis, lineis, &c. quando maximæ vel minimæ ſint.

EXEMPLUM I.

Sit ovalis, nullum habens contrariæ flexuræ punctum, & ducantur Fig. 56. (n) lineæ $(EF, FG, GH, HI,$ &c.) ovalem tangentes, in (n) punctis $(A, B, C, D,$ &c.) ſint etiam hæ lineæ ad contactuum puncta $(A, B, C, D,$ &c.) in duas æquales partes diviſæ; i. e. $EA = AF$, $FB = BG$, $GC = CH$, $HD = DI$, &c. tum minus erit polygonum $(EFGH,$ &c.) quam quodvis aliud, idem contactûs punctum (A) habens, & circa datam ovalem deſcriptum.

Si enim circumſcriptum polygonum, cujus latera $EF, FG, GH,$ &c. in duas æquales partes, in punctis contactuum $(A, B, C, D,$ &c.) dividuntur, non ſit minimum, quod circa ovalem deſcribi poteſt, & cujus punctum contactûs eſt (A); ſit polygonum $(LMNR,$ &c.) miminium, habens latus (MN) a puncto contactûs (a) in duas inæquales partes aM & aN diviſum, quarum (aM) major ſit quam aN.

Ducatur linea $\alpha\beta$, lineas LM & NR in punctis α & β ſecans, & in duas æquales partes $(\alpha\gamma = \gamma\beta)$ ad punctum (γ) contactûs diviſa: tum polygonum $(L\alpha\beta NR,$ &c.) minus eſt quam polygonum $(LMNR,$ &c.) quoniam triangulum $Mb\alpha$ minus eſt quam triangu-

 lum

lum $Nh\beta$; & conſequenter polygonum (*LMNR*, &c.) non eſt minimum quod circumſcribi poteſt, niſi ejus ſingulum latus in duas æquales partes *a* puncto ejus contactus dividitur.

EXEMPLUM II.

Fig. 57. *Sit eadem ovalis, & in eâ inſcribatur polygonum* (n) *laterum* (*A*,*B*,*C*,*D*), &c. per ſucceſſiva puncta (*A*, *B*, *C*, *D*, &c.) *ducantur tangentes* (EF, FG GH, *&c.*) *reſpective, & ſit linea* AC *parallela lineæ tangenti ovalem in puncto* (B), *linea* BD *parallela lineæ ovalem tangenti in puncto* (C), *& ſic deinceps: tum polygonum* (ABCD, *&c.*) *erit maximum polygonum* (n) *laterum, quod in datâ ovali inſcribi poteſt, & cujus punctum contactus eſt* (A).

Si enim lineæ *AC*, B*D*, *CE*, &c. non ſunt reſpective parallelæ lineis tangentibus ovalem in punctis (*A*, *B*, *C*, *D*, &c.) ſit *BD* haud parallela tangenti in puncto (*C*), & lineæ (*BD*) ducatur linea $\alpha\gamma$ parallela, tangens circulum in puncto (α); tum triangulum $B\alpha D$ majus erit quam triangulum *BCD*; & conſequenter polygonum *ABCD*, &c.) non eſt maximum polygonum (*n*) laterum, quod in datâ ovali inſcribi poteſt, habens ejus verticem punctum (*A*), ni prædictæ lineæ ſunt parallelæ reſpectivis tangentibus.

EXEMPLUM III.

Fig. 57. *Iiſdem poſitis, inſcribatur polygonum* (n) *laterum* (ABCDE, &c.) *& per ſucceſſiva puncta* (ABCDE, &c.) *ducantur lineæ* EF, FG, GH, HI, *&c. ovalem tangentes, in punctis* (A, B, C, D, *&c.*) *& ſit angulus* ABF $=\angle$CBG, *& angulus* BCG $=\angle$DCH, *&* CDH $=$ EDI, *&c. i. e. anguli, quos latera polygoni utrinque cum reſpectivis tangentibus ad contactuum puncta faciunt, ſint inter ſe æquales.*

Et ſumma polygoni (*n*) laterum erit major, quam ſumma, cujuſcunque alius polygoni (*n*) laterum in ovali inſcripti, & cujus vertex eſt punctum (*A*).

EXEMPLUM IV.

Fig. 57. *Deſcribatur circa ovalem polygonum* n *laterum* (EFGH, *&c.*) *cujus punctum contactus eſt* (A), *ſint puncta contactuum reſpective* (A, B, C, D, *&c.*) *& ſit*

A

AE : AF : : $\overline{\text{Tan} + \text{Sec}}$ Com : ∠ AFG : $\overline{\text{Tan} + \text{Sec}}$ Com : ∠ AEP
FB : BG : : $\overline{\text{Tan} + \text{Sec}}$ Com : < BGC : $\overline{\text{Tan} + \text{Sec}}$ Com : < $\overline{\text{BFA}}$
GC : CH : : $\overline{\text{Tan} + \text{Sec}}$ Com : < CHD : $\overline{\text{Tan} + \text{Sec}}$ Com : < CGB
& ſic deinceps, tum.

Summa (*n*) laterum polygoni, minor erit quam ſumma (*n*) laterum cujuſcunque alius polygoni, circa ovalem deſcripti, & cujus punctum contactus eſt (*A*).

EXEMPLUM V.

Deſcribatur circa ovalem, polygonum (n) *laterum* (EFGH, &c.) *cujus puncta contactuum ſint* (A, B, C, D, *&c.*) *in ſingulam tangentem ducantur duo perpendicula a duobus proximis angulis, contactus puncto utrinque jacentibus, i. e. ducantur* AP & CQ *perpendicula in tangentem* EBF; BR & DS *perpendicula in tangentem* FBG, *& ſic deinceps.* Fig. 57.

Sit etiam

$$AB^{m-2} : BC^{m-2} : : BQ : BP$$

& $BC^{m-1} : DC^{m-2} : : CS : CR$ & ſic de reliquis, tum ſumma $AB^m + BC^m + CD^m + DE^m +$ &c. major erit quam ſumma (*m*) poteſtatis ex ſingulis (*n*) lateribus cujuſcunque alius polygoni in ovali inſcripti, cujus punctum contactus eſt (*A*).

EXEMPLUM VI.

Iiſdem poſitis, ſit Fig. 57.

$$AB^2 : BC^2 : m \times PB : o \times BQ \text{ \&}$$
$$BC^2 : CD^2 : : o \times CR : p \times CS \text{ \& ſic}$$

deinceps, tum contentum

$$AB^m \times BC^o \times CD^p \times \text{\&c.}$$

majus erit, quam quodcunque aliud ſimile contentum, cujus punctum contactus eſt (*A*).

Et ſic de infinitis ſimilibus exemplis.

3. Si autem ad omne curvæ punctum referuntur lineæ, curvæ, &c. inter quas requiruntur æquationes relationem exprimentes.

Inveniantur linearum, arearum, &c. incrementa; & exinde erui poſſunt æquationes quæſitam relationem inter eas lineas, areas, &c. exprimentes.

EXEMPLUM I.

Circa ovalem, nullum habentem contrariæ fluxuræ punctum, deſcribantur duo polygona (n) *laterum* EFGH, &c. & PQRS, &c. & *ita ut utriuſque* Fig. 58.

uſque polygoni latera ſecentur in duas æquales partes in punctis contactuum A, B, C, D, &c; & α, β, γ, δ, &c; *reſpective*; *tum polygonum* (E F G H) &c. *erit æquale polygono* (P Q R S &c.)

DEMONSTRATIO.

Hæc polygona ſunt minima, quæ a reſpectivis punctis (*A* & α) incipiunt, & circa ovalem deſcribi poſſunt.

Deſcribantur etiam polygona, incipientia a punctis inter (*A* & *a*) poſitis, & ita ut ſint minima, quæ a reſpectivis punctis incipiunt.

Tum, quoniam polygonum perpetuo eſt minimum, quod deſcribi poteſt circa ovalem, nullum habet incrementum, & conſequenter inter ſe æqualia perpetuo manent polygona, & exinde ſequitur polygonum (*E F G H* &c.) æquale eſſe polygono (*P Q R S* &c).

COR. I.

Fig. 59. Deſcribatur circa ovalem triangulum (*EFG*), cujus latera (*EF*, *FG*, *EG*) reſpective dividuntur in æquales partes ($AE = AF$, $BF = BG$, $CG = CE$) in punctis contactuum (*A*, *B*, *C*), *i. e.* ſit triangulum (*EFG*) minimum, quod circa ovalem deſcribi poteſt, & jungantur puncta contactuum (*A*, *B*, *C*), & triangulum (*ABC*) erit maximum, quod in datâ ovali inſcribi poteſt.

Facile enim conſtat lineam (*AB*) parallelam eſſe tangenti (*EG*), lineam (*BC*) parallelam tangenti (*EF*), & lineam (*AC*) tangenti (*FG*); & conſequenter triangulum (*ABC*) maximum eſſe, quod in datâ ovali deſcribi poteſt.

E lemmate primo conſtat maximum triangulum, quod in datâ ovali inſcribi poteſt, eſſe ad minimum triangulum, quod circumſcribi poteſt, in ratione unitatis ad numerum quaternarium.

COR. II.

Fig. 60. Deſcribatur circa ovalem trapezium (*EFGH*), cujus latera in æquales partes per ſucceſſiva contactus puncta (*A*, *B*, *C*, *D*) dividuntur, jungantur hæc contactus puncta, & Trapezium (*ABCD*) exinde formatum erit parallelogrammum, & ejus area dimidium areæ circumſcripti Trapezii.

Hoc corollarium ſequitur e lemmate ſecundo.

CO-

COR. III.

Sit ovalis, conici generis ellipſis, & circa conjugatas ejus diametros (*AB* & *CD*) deſcribatur parallellogrammum (*EFGH*); hujus parallelogrammi latera in punctis contactuum (*A*, *B*, *C*, *D*) biſecantur in duas æquales partes, & conſequenter minimum eſt trapezium, quod circa ovalem deſcribi poteſt, unde omnia parallellogramma circa conjugatas diametros ellipſeos deſcripta, erunt inter ſe æqualia. Fig. 60.

Et ſic circumſcribantur circa ellipſim polygona (*EFGH* &c. *PQRS* &c.) (*n*) laterum, a diverſis contactus punctis (*A* & *a*) incipientia, & ſint reſpective minima, quæ circa ellipſim deſcribi poſſunt, a punctis (*A* & *a*) incipientia; tum polygonum (*EFGH* &c.) erit æquale polygono (*PQRS* &c.) & minimum circumſcriptum polygonum (*EFGH* &c.) habet ad maximum inſcriptum eandem rationem, quam habet quadratum radii ad quadratum coſinus arcus $\left(\frac{360}{2n}\right)$. Fig. 58.

EXEMPLUM II.

Iiſdem poſitis, in ovali inſcribantur duo (*n* laterum) polygona (*ABCDEF*, &c. & $\alpha\beta\gamma\delta\epsilon$, &c.) ita ut diagonalis *AC* parallela ſit lineæ tangenti ovalem in puncto contactus (*B*), & ſic diagonales *BD*, *CE*, *DF*, &c. parallelæ ſint reſpective lineis curvam tangentibus in punctis contactuum *C*, *D*, *E*, &c. & ſimiliter ſit diagonalis $\alpha\gamma$ parallela lineæ tangenti ovalem in puncto (β), & lineæ $\beta\delta$, $\gamma\epsilon$, &c. ſint parallelæ lineis tangentibus ovalem in punctis (γ, δ, &c.). Fig. 61.

Hæc duo polygona *ABCDE*, &c. $\alpha\beta\gamma\delta\epsilon$, &c. erunt inter ſe æqualia.

Hoc exemplum eadem methodo, quâ præcedens, demonſtrari poteſt.

COROLLARIUM I.

Sit polygonum, triangulum (*ABC*), circumſcribatur triangulum (*EFG*), cujus puncta contactus ſunt (*A*, *B*, *C*), & triangulum (*EFG*) erit minimum, quod circumſcribi poteſt, & inſcriptum erit ad circumſcriptum: in ratione unitatis ad numerum quaternarium. Fig. 59.

COROLLARIUM II.

Sit polygonum trapezium (*ABCD*), & circumſcriptum trapezium, cujus puncta contactuum ſint (*A*, *B*, *C*, *D*), erit parallelogrammum, & Fig. 60.

habet

habet ad inſcriptum ($ABCD$), eandem rationem, quam habet numerus binarius ad unitatem.

COROLLARIUM III.

Sint duo trapezia ($ABCD$, & $\alpha\beta\gamma\delta$) maxima, quæ in datâ ovali inſcribi poſſunt, a punctis (A & α) incipientia, & ſit trapezium $ABCD$ parallelogrammum, tum etiam erit trapezium $\alpha\beta\gamma\delta$ parallelogrammum. Et ſic de circumſcriptis.

COROLLARIUM IV.

Sit ovalis ellipſis conici generis, circumſcribatur circa eam polygonum (n) laterum, cujus puncta contactuum ſunt reſpective ($ABCD$, &c.) & erit circumſcriptum minimum, quod circa ellipſim deſcribi poteſt.

EXEMPLUM III.

Fig. 61. *Iiſdem poſitis, in ovali duo polygona* (ABCDE, *&c.* & $\alpha\beta\gamma\delta$*&c.*) *inſcribantur, ita ut lineæ* AB, BC; CD, DE, *&c. æquales angulos cum lineis ovalem tangentibus in punctis contactuum* (A, B, C, D, *&c.*) *utrinque faciant, & ſic lineæ* $\alpha\beta$, $\beta\gamma$; $\gamma\delta$, $\delta\epsilon$, *&c. æquales angulos cum lineis tangentibus ovalem in punctis contactuum* $\alpha, \beta, \gamma, \delta, \epsilon$, *&c. tum ſummæ polygonorum laterum erunt reſpective inter ſe æquales, i. e.*

$$AB+BC+CD+DE, \&c. = \alpha\beta+\beta\gamma+\gamma\delta+\delta\epsilon+ \&c.$$

COROLLARIUM I.

Fig. 59. Sit polygonum triangulum (ABC), & per puncta contactuum ($A, B, C,$) ducantur tangentes (EF, FG, GE), & ſumma laterum inſcripti trianguli (ABC) non habet ad ſummam laterum circumſcripti trianguli (EFG) minorem rationem quam $1 : 2$.

COR. II.

Fig. 60. Sit polygonum, trapezium ($ABCD$), & ducantur methodo prædicta tangentes (EF, FG, GH, HE), & trapezium circumſcriptum $EFGH$ non habet ad trapezium inſcriptum $ABCD$ majorem rationem quam $\sqrt{2} : 1$.

COR. III.

Circa trapezium ($EFGH$) ſemper deſcribi poteſt circulus, qui tranſit per puncta E, F, G, H reſpective.

COR.

COROLLARIUM IV.

Sit ellipſis conici generis *ABCD*, cujus duo foci ſunt *F* & *H*; & lineæ a duobus focis ductæ ad quodlibet peripheriæ punctum (*A*), eundem angulum utrinque cum tangente, cujus punctum contactus eſt (*A*), faciunt; & conſequentei per hoc exemplum ſemper eadem manebit harum duarum linearum ($HA+AF$) ſumma; i.e. ſit *B* quodlibet aliud ellipſeos peripheriæ punctum, & Fig. 62.

$AH+AF=BH+BF$; & vicem verſam æquo jure concludere licet.

EXEMPLUM IV.

Sint duo polygona (EFGHI, *&c.* & PQRS, *&c.*) (*n*) *laterum, circa ovalem deſcripta, & ſint puncta contactus prioris polygoni reſpective* A, B, C, D, *&c. puncta vero poſterioris* α, β, γ, δ, *&c.* & Fig. 63.

$$FA : AE :: \overline{\text{Tan}+\text{Sec}}\ \text{Com} \angle AEp : \overline{\text{Tan}+\text{Sec}}\ \text{Com} \angle AFG$$

$$FB : GB :: \overline{\text{Tan}+\text{Sec}}\ \text{Com} \angle BGC : \overline{\text{Tan}+\text{Sec}}\ \text{Com} \angle BFA$$

& *ſic deinceps.*

Et eodem modo ſecentur polygoni *PQRST*, &c. latera in punctis contactuum α, β, γ, δ, &c. & ſumma laterum

$$FE+FG+GH+ \&c. = PQ+QR+RS+ST+ \&c.$$

COROLLARIUM I.

Sit polygonum triangulum (*EFG*), & jungantur puncta contactuum *A*, *B*, *C*; & ſumma laterum trianguli *ABC*, i.e. $AB+BC+AC$ non poteſt habere ad ſummam laterum trianguli (*EFG*), $EF+FG+EG$ minorem rationem quam 1 : 2.

Sit polygonum trapezium *EFGH*, & jungantur puncta contactus *A*, *B*, *C*, *D*; & non poteſt habere ſumma laterum $EF+FG+GH+HE$ ad ſummam laterum $AB+BC+CD+DA$ majorem rationem quam $\sqrt{2} : 1$.

EXEMPLUM V.

Inſcribantur duo polygona n *laterum in ovali* (ABCD, *&c.* αβγδ, *&c.*) *ita ut contentum ſub ſingulis* (n) *lateribus* AB×BC×CD× *&c. majus ſit quam contentum ſub* (n) *lateribus cujuſcunque alius polygoni, cujus vertex eſt* (A); & *contentum* αβ×βγ×γδ× *&c. majus ſit quam contentum ſub*

(n) *lateribus omnis alius polygoni, cujus vertex eſt* (α); *tum hæc duo contenta erunt inter ſe æqualia; i. e.* AB×BC×CD×&c. = αβ×βγ×γδ×&c.

EXEMPLUM VI.

Sit ſumma (m) *poteſtatis ex ſingulis* (n) *lateribus polygoni*, (ABCD, &c.) *major quam ſumma* (m) *poteſtatis ex ſingulis* (n) *lateribus cujuſcunque alius polygoni eundem verticem habentis; & ſimiliter ſit ſumma* (m) *poteſtatis ex ſingulis* (n) *lateribus polygoni* (αβγδ, &c.) *major quam ſumma* (m) *poteſtatis ex ſingulis* (n) *lateribus omnis reliqui polygoni eundem verticem habentis, tum hæ duæ ſummæ erunt inter ſe æquales, i. e.*

$$AB^m + BC^m + CD^m + \&c. = \alpha\beta^m + \beta\gamma^m + \gamma\delta^m + \&c.$$

Et ſic progredi licet ad polygona hujuſmodi circumſcripta.

EXEMPLUM VII.

Fig. 64. *Sit ſegmentum curvæ* ACB, *& ducatur tangens* ECF *ad ſegmentum in quolibet puncto* C, *ſit angulus* BCF *ſemper æqualis angulo in alterno ſegmento* BAC, *tum erunt anguli in ſegmento contenti* BCA *vel* BDA *ſemper inter ſe æquales.*

Si enim angulus *BCF* æqualis ſit angulo *BAC* in alterno ſegmento, tum erit angulus in ſegmento (*ACB*) maximus; & conſequenter ejus incrementum nihilo eſt æquale, ſi vero incrementum anguli in dato ſegmento contenti continuo nihilo eſt æquale, tum iſte angulus ſemper idem manet i. e. $\angle BAC = \angle BDC$.

Et ſic de infinitis hujuſmodi proportionibus.

EXEMPLUM VIII.

Inſcribantur duæ ellipſes in datâ ovali, quæ nullum habet contrariæ flexuræ punctum, & incipientes a quibuſcunque diverſis ovalis punctis, & ſint ellipſes maximæ, quæ ab iſtis punctis inſcribi poſſunt; tum hæ duæ ellipſes erunt æquales inter ſe.

Et ſic de pleriſque aliis ovalibus.

EXEMPLUM IX.

Sit ſolidum, in ſeſe omni parte concavum, & inſcribantur coni, quorum vertices ſunt quæcunque dati ſolidi puncta, & ſint hi coni maximi, qui pro

pro verticibus ſuis reſpectiva puncta habent; tunc omnes hi coni erunt inter ſe æquales.

Omnes hæ propoſitiones æque applicari poſſunt in curvas, quæ ſibi ipſis in omni parte ſunt convexæ.

In genere inſcribantur vel circumſcribantur lineæ, ita ut ſolida, vel areæ, vel lineæ ſint maxima, vel quandam inter ſe habeant datam relationem; & mutetur punctum contactus, & continuo ſint ſolida, vel areæ, vel lineæ, maxima, vel habeant inter ſe relationem, quam exigat problema; tum ſolida, areæ, vel lineæ, utrum ab hoc, vel ab iſto puncto incipiunt, erunt inter ſe æqualia, vel requiſitam habent inter ſe relationem.

Ex hypotheſi enim ſolida, areæ, vel lineæ, ſemper ſunt maxima, &c. & conſequenter eorum incrementa nihilo ſunt æqualia, &c. ergo ſolida, areæ, vel lineæ continuo inter ſe manent æqualia, &c.

Facile conſtabit quomodo propoſitiones hujuſce generis applicari poſſunt in curvas, quæ in aliâ parte ſunt concavæ, in aliâ vero convexæ.

4. Invenire relationem inter lineas diverſarum curvarum.

Datis æquationibus, relationem inter quaſdam datarum curvarum lineas, exprimentibus, inveniatur relatio inter has lineas & lineas, inter quas requiritur relatio, quibus æquationibus reductis, reſultant æquationes relationem quæſitam exprimentes.

Hujus caſus facile eſt adjicere exempla.

CAP. V.

OMNES curvæ tertii generis reduci poſſunt ad duodecim ſubſequentes æquationum caſus.

CAS. 1.

Sit æquatio $y = Ax^4 + Bx^3 + Cx^2 + Dx + E$, & hæc æquatio deſignat biquadraticam parabolam diametrum habentem, quæ eſt prima ſpecies.

CAS. 2.

Sit æquatio $y^3 + \overline{c + dx + ex^2}\,xy = Ax^4 + Bx^3 + Cx + Dx + E$, & hæc æquatio deſignat parabolam tertii generis.

Curvarum ſpecies haud plures ſunt, quam vel pura parabola; vel parabola cum ovali, vel puncto, intus vel extra parabolam, poſitis; vel cum duabus ovalibus; vel ovali & puncto; vel ovali nodatâ, vel cuſpidatâ; vel duobus punctis; vel cum tribus ovalibus; vel duabus ovalibus & puncto, vel nodo, vel cuſpide; vel unâ ovali & duobus punctis, vel nodo & puncto, vel duobus nodis, vel cuſpide & puncto, vel cuſpide & nodo, vel duabus cuſpidibus; vel parabola cum tribus punctis; vel nodo, qui intus vel extra parabolam flectitur; vel parabola nodum vel cuſpidem habens, & cum ovali, vel puncto, vel duabus ovalibus, vel ovali & puncto; vel ovali nodata; vel cuſpidatâ; vel duobus punctis; vel parabola habet duos nodos ad idem punctum, vel duas cuſpides, vel nodum & cuſpidem ad idem vel diverſa curvæ puncta, & prætere a habere poteſt ovalem vel punctum; vel parabola habet duos nodos ad diverſa curvæ puncta, vel nodum & duas cuſpides, quarum una vel ponitur ad nodum, necne; vel duos nodos ad idem curvæ punctum cum cuſpide vel ad nodum, necne; vel tres nodos ad idem punctum, vel tres cuſpides; vel parabola habet ovalem curvam interſecantem in duobus punctis; vel punctum triplex; ergo ſpecierum numerus haud major eſt quam 59.

Sit æquatio $y^4 + \overline{a + bx}\,y^2 = Ax^3 + Bx^2 + Cx + D$, & denotat parabolas tertii generis diametrum habentes.

Earum ſpecies ſunt, vel pura parabola, vel parabola cum ovali vel puncto vel nodo intus vel extra parabolam poſitis, vel cuſpide, vel duabus

duabus ovalibus, vel ovali nodum habente, vel duobus punctis, vel duobus nodis ad curvam, vel triplici nodo, vel duabus cuſpidibus, vel ovali vel puncto vel ad curvam nodo vel cuſpide cum duabus ovalibus vel punctis, vel puncto triplici; ergo totus ſpecierum hujus caſus numerus haud major eſt, quam $59+23=82$.

CAS. 3.

1. Sit æquatio
$y^3+\overline{a+bx+cx^2}\times y^2+\overline{d+ex+fx^2+gx^3}\times y=Ax^4+Bx^3+Cx^2+Dx+E$, in quâ radices æquationis $cy^2+gy=A$, ſunt impoſſibiles, & denotat curvas unam parabolam conici generis habentes.

Numerus ſpecierum curvarum hujuſce caſus ſine diametro non eſt major, ut conſtat e præcedenti caſu, quam 59.

2. Sit æquatio $y^4+\overline{cx^2+dx+e}\times y^2=Ax^3+Bx^2+Cx+D$ & denotat parabolas conici generis, diametrum habentes.

Earum ſpecies ſunt, vel pura parabola, vel parabola cum ovali vel puncto vel nodo intus vel extra parabolam poſitis, vel cuſpide, vel duabus ovalibus, vel ovali nodum habente, vel duobus punctis, vel duobus nodis ad curvam, vel triplici nodo, vel duabus cuſpidibus; vel cum tribus ovalibus, vel duabus ovalibus vel punctis, & puncto vel ad curvam nodo vel cuſpide; vel ovali & duobus punctis vel duobus nodis, vel ad curvam duobus nodis vel cuſpidibus; vel duobus nodis vel cuſpidibus ad curvam cum puncto; vel puncto triplici; ergo numerus ſpecierum haud major eſt quam $59+28=87$.

CAS. 4.

1. Sit æquatio $y^2+\overline{a+bx+cx^2}\times y=Ax^4+Bz^3+Cx^2+Dx+E$; & ſint radices æquationis $y^2+cy=A$ poſſibiles; & æquatio deſignat curvas duas parabolas conici generis habentes.

Harum curvarum ſpecies ſunt; vel duæ puræ curuæ, quæ duas conſtituunt ſpecies, vel duæ curvæ ſeſe inter ſecant figurâ cruciformi, vel habent ovalem vel punctum vel nodum vel cuſpidem vel duos nodos curvas jungentes, quarum ſpecierum numerus eſt 8, ſed crura in infinitum eâdem directione pergentia, vel ſunt convexa vel concavo-convexa inter ſe; ergo numerus ſpecierum eſt 16.

Si deſint termini $\overline{a+bx+cx^2}\times y$, curvæ habent diametrum, & numerus ſpecierum, ut conſtat e prædictis, eſt 8.

Sit

Sit æquatio $y^2+\overline{a+cx^2}\times y=Ax^4+Cx^2+E$, & curvæ habent diametrum; earum ſpecies ſunt vel duæ puræ curvæ, quæ duas conſtituunt ſpecies, vel ſeſe interſecant figurâ cruciformi, vel hæ duæ curvæ habent inter eas ovalem vel puuctum vel duos nodos curvas jungentes, quarum numerus eſt 6, ſed crura eâdem directione in infinitum pergentia ſunt convexa vel concavo-convexa inter ſe; ergo numerus ſpecierum eſt 12.

Sit æquatio $y^2=Ax^4+Cx^2+E$, & curvæ habent duas diametros; & ſpecies ſunt, vel duæ puræ curvæ, quæ duas conſtituunt ſpecies, vel hæ duæ curvæ ſeſe interſecant figurâ cruciformi, vel habent ovalem vel punctum, quarum ſpecierum numerus eſt 5, & totus numerus $16+8+12+5=41$.

2. Sit æquatio

$$y^3+\overline{a+bx+c^2x^2}\times y^2+\overline{d+ex+fx^2\pm 2cgx^3}\times y=-g^2x^4+Bx^3+Cx^2+Dx+E$$

& haud plures poſſunt eſſe ſpecies, quam vel duæ puræ parabolæ conici generis, vel duæ parabolæ cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto vel nodo vel cuſpide; vel duobus punctis; vel cum tribus ovalibus, vel duabus ovalibus & puncto vel nodo vel cuſpide; vel ovali cum duobus nodis, vel nodo & puncto, vel duobus punctis, vel nodo vel puncto & cuſpide, vel duabus cuſpidibus, vel tribus punctis; vel parabola habet nodum vel cuſpidem, vel habet nodum vel cuſpidem cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto, vel ovali nodatâ vel cuſpidatâ, vel duobus punctis; vel parabola habet duos nodos ad idem punctum, vel nodum & cuſpidem ad eandem vel diverſas curvas, vel duas cuſpides ad diverſas curvas, vel etiam cum ovali vel puncto; vel duos nodos ad diverſas curvas, vel duos nodos, vel nodum & cuſpidem ad unam; & nodum vel cuſpidem ad alteram; harum ſpecierum numerus eſt 52; ſed ſingula ex his ſpeciebus diſtingui poteſt in duas alias, prout duæ parabolæ ſunt disjunctæ, vel unius crus adjungitur alterius cruri; & conſequenter ſpecierum numerus eſt 104; ſed etiam hæ parabolæ ſeſe interſecare poſſunt cruciformiter, vel cruciformiter cum ovali, vel puncto, vel nodo, vel cuſpide, vel cum duabus ovalibus, vel ovali & puncto vel nodo vel cuſpide; vel duobus punctis, vel cum nodo vel cuſpide ad curvam etiamque cum ovali vel puncto vel ad

hanc

hanc vel iſtam curvam nodo vel cuſpide, quarum ſpecierum numerus eſt 36, vel curvæ bis ſeſe interſecare poſſunt, vel bis, cum ovali, vel puncto, vel nodo vel cuſpide ad curvam, vel ter ſeſe interſecare poſſunt; unde totus ſpecierum numerus eſt 193.

CAS. 5.

1. Sit æquatio $\overline{x^2+2ax+a^2+b^2}\times y^2+\overline{2cx^3+dx^2+ex+f}\times y = \overline{-c^2+A^2}x^4+Bx^3+Cx^2+Dx+E$; & hæc æquatio deſignat curvas, quarum nullum crus in infinitum progreditur.

Earum ſpecies ſunt; vel ovalis; vel duæ ovales; vel ovalis & punctum; vel ovalis nodum vel cuſpidem habens; vel tres ovales, vel duæ ovales & punctum vel nodus; vel duæ ovales, quarum una habet cuſpidem; vel ovalis cum duobus punctis, vel nodo & puncto, vel duobus nodis; vel ovalis nodum vel punctum & cuſpidem habens; vel ovalis cum duabus cuſpidibus, vel puncto triplici; quarum ſpecierum numerus eſt 16.

2. Sit æquatio $y^4+\overline{c+dx+ex^2}\times y^2+\overline{f+gx+bx^2+kx^3}\times y = Ax^4+Bx^3+Cx^2+Dx+E$, in quâ omnes radices æquationis $y^4+ey^2+ky=A$ ſunt impoſſibiles; & æquatio deſignat curvas, quarum nullum crus in infinitum progreditur, & quarum ſpecies eædem ſunt ac præcedentes cum quatuor ſubſequentibus; vel una ovali in altera incluſa, vel ovali punctum vel nodum vel duos nodos in ſe continente; quarum ſpecierum numerus eſt 20.

Sit æquatio $P\overline{x^2+a^2}\times y^2 = -A^2x^4+Bx^3+Cx^2+Dx+E$, & deſignat curvas diametrum habentes.

Earum ſpecies ſunt, vel una ovalis; vel duæ; vel punctum & ovalis; vel ovalis cum nodo vel cuſpide.

Sit æquatio $P\times\overline{x^2+a^2}\times y^2 = -A^2x^4+Cx^2+E$, & deſignat curvas duas diametros habentes, quarum ſpecies ſunt, vel una ovalis; vel duæ, vel ovalis nodata.

Sit æquatio $y^4+\overline{a+2cx^2}\times y^2 = -\overline{A^2+c^2}+x^4+Cx^2+E$, & deſignat curvas duas diametros habentes. quarum ſpecies ſunt eædem ac præcedentes cum tribus ſubſequentibus, vel ovali continente in ſe ovalem vel punctum; vel duabus ovalibus ſeſe in duobus punctis ſecantibus.

Sit

Sit æquatio $y^4+\overline{a+bx+cx^2}\times y^2=-\overline{A^2+c^2}\times x^4+Bx^3+Cx^2+Dx+E$, & deſignat curvas unam diametrum habentes, quarum ſpecies ſunt, vel ovalis, vel duæ ovales, vel ovalis & punctum, vel ovalis nodum vel cuſpidem habens; vel tres ovales, vel duæ ovales cum puncto, vel nodo; vel ovalis nodum & punctum, vel duos nodos, vel duas cuſpides, vel punctum triplex, vel duo puncta, habens; vel etiam ovalis habens in ſe ovalem, vel punctum, vel ovalis intus ſemel vel bis nodata, ergo totus ſpecierum numerus hujus caſus eſt $20+6+17=43$.

CAS. 6.

Sit æquatio $x^2y^2+\overline{ax^2+bx+c}\times y=-Ax^4+Bx^3+Cx^2+Dx+E$; & deſignat circumſcriptam hyperbolam unam aſymptoton habentem.

Si $-Ax^6+Bx^5+Cx^4+Dx^3+Ex^2+\frac{1}{4}\times\overline{ax^2+bx+c}^2=0$; tum duo ordinatæ ($y$) valores erunt inter ſe æquales.

Et exinde innoteſcunt ſpecies; quæ ſunt vel pura hyperbola; vel hyperbola cum ovali; vel puncto; vel nodo; vel cuſpide; vel cum duabus ovalibus; vel ovali & puncto; vel ovali & nodo; vel ovali nodum vel cuſpidem habente; vel cuſpide & ovali vel puncto; vel duobus punctis; vel puncto & nodo; vel duobus nodis ad idem vel diverſa curvæ puncta; vel duabus cuſpidibus; vel nodo & cuſpide ex eadem vel diverſa aſymptoti parte; quarum ſpecierum numerus eſt 19.

Si terminus (cy) deeſt; tum figura conſtat ex duabus hyperbolis ex eadem vel diverſis aſymptotos partibus utrinque in infinitum progredientibus; & ſpecies erunt, vel duæ puræ hyperbolæ, vel duæ hyperbolæ cum ovali; vel puncto; vel nodo; vel cuſpide; quæ ſunt 10 ſpecies.

Si $x^2y^2+\overline{ax^2+c}\times y=-Ax^4+cx^2+E$; æquatio deſignat circumſcriptam hyperbolam diametrum habentem; ejus ſpecies ſunt, vel pura circumſcripta hyperbola; vel hyperbola cum duabus ovalibus; vel duobus punctis; vel duobus nodis; vel duabus cuſpidibus; quæ ſunt 5 ſpecies.

Si terminus (cy) deſit, tum duæ ſunt hyperbolæ circa eandem aſymptoton utrinque in infinitum pergentes.

Si

Si $y^2x^2=-Ax^4+Bx^3+Cx^2+Dx+E$, deſignat duas hyperbolas diametrum habentes circa eandem aſymptoton circumſcriptas; & earum ſpecies ſunt, vel duæ puræ hyperbolæ; vel duæ hyperbolæ cum ovali, vel puncto, vel nodo, vel cuſpide; quæ ſunt 5 ſpecies.

Sit æquatio $x^2y^2=-Ax^4+Cx^2+E$, & duæ hyperbolæ duas habent diametros; ergo totus numerus ſpecierum hujuſce caſus eſt 41.

CAS. 7.

1. Sit æquatio $xy=Ax^4+Bx^3+Cx^2+Dx+E$; & figura conſtat ex duabus curvis, quæ partim hyperbolicæ, partim cubico-parabolicæ ſunt.

2. Sit æquatio $y^2+\overline{a+bx+cx^2+dx^3}\times y=Ax^2+Bx+C$, & figura vel conſtat e duabus curvis partim hyperbolicis & partim cubico-parabolicis, quarum hyperbola & parabola continuo ductu vel in oppoſitas partes progrediuntur, vel in eandem redit, & hyperbolicum crus vel eſt concavum vel convexum in parabolicum verſus eandem partem pergens, quæ quatuor conſtituunt ſpecies; vel conſtat e duabus hyperbolo-parabolicis curvis, quarum hyperbolica & parabolica crura in eandem partem pergentia vel ſunt convexa vel concavo-convexa inter ſe cum figurâ cruciformi; vel ovali; vel puncto; vel nodo; vel cuſpide; vel duabus ovalibus; vel ovali nodum habente; vel ovali & nodo ad curvam; vel ovali & puncto vel cuſpide ad ovalem vel curvam; vel cuſpide ad unam curvam & nodo ad alteram; vel cuſpide & nodo ad eandem; vel cuſpide & puncto; vel duobus nodis ad eandem vel diverſas curvas; vel puncto & nodo ad curvam; vel duobus punctis; vel duobus vel tribus nodis duas curvas connectentibus; vel duabus cuſpidibus; ſi terminus Ax^2 deeſt, tum crura hyperbolica ex eodem aſymptoti latere ad infinitam diſtantiam jacent, & conſtat ſpecierum numerum minorem eſſe per unitatem quam numerum curvarum, quarum hyperbolica crura verſus eandem partem progredientia ſunt convexa vel concavo-convexa inter ſe; quoniam non poteſt habere duas cuſpides curva, cujus æquatio eſt prædictæ formulæ, majorem vero per 9, quoniam duæ curvæ diverſæ ſunt ſpeciei inter ſe; ergo totus numerus ſpecierum hujus caſus eſt 77.

CAS. 8.

1. Sit æquatio $xy^2 + ay = Ax^4 + Bx^3 + Cx^2 + Dx + E$ & deſignat ſemicubicam parabolam cum hyperbolâ.

Curvarum hujus æquationis ſpecies ſunt, vel duæ curvæ, quæ partim parabolicæ & partim hyperbolicæ ſunt; vel hæ duæ curvæ cum ovali vel puncto vel nodo vel cuſpide, vel duabus ovalibus, vel ovali & puncto; vel ovali & ad curvam nodo vel cuſpide; vel ovali nodum vel cuſpidem habente; vel duobus punctis, vel puncto & ad curvam nodo vel cuſpide; vel ad curvam duobus nodis, vel nodo & cuſpide: ſpecies etiam erunt vel ſemicubica parabola cum anguineâ hyperbolâ, vel ſemicubica parabola & anguinea hyperbola cum ovali, vel puncto, quæ vel ponuntur inter parabolam & hyperbolam, necne; vel nodo vel cuſpide ad parabolam; vel ad hyperbolam nodo vel cuſpide ex hac vel illâ aſymptoti parte; vel parabola & hyperbola ſeſe interſecant, vel ſeſe interſecant & habent ovalem vel punctum, vel ad hyperbolam nodum vel cuſpidem, vel hyperbola & parabola bis ſeſe interſecant, ergo ſpecierum numerus eſt 33.

Si terminus (ay) deſit, tum curvæ habent diametrum, & ſpecies ſunt vel duæ puræ curvæ partim parabolicæ & partim hyperbolicæ, vel hæ duæ curvæ cum ovali; vel puncto; vel duabus ovalibus; vel ovali & puncto; vel ovali nodum vel cuſpidem habente; vel duobus punctis; vel ſint parabola & conchoidalis hyperbola ex eâdem vel diverſâ aſymptoti parte, vel hæ duæ curvæ cum ovali vel puncto inter eas, necne, vel ad parabolam vel hyperbolam nodo vel cuſpide, vel parabola & hyperbola ſeſe interſecant, vel ſeſe interſecant & habent ovalem vel punctum, vel parabola & hyperbola bis ſeſe interſecant, ergo harum ſpecierum numerus eſt 30.

2. Sit æquatio $y^3 + \overline{a + bx + cx^2 + dx^3} \times y = Ax^2 + Bx + C$, & deſignat duas curvas partim ſemicubicas parabolas & partim hyperbolas; in quibus hyperbolicum crus & ejus correſpondens parabolicum in infinitum eâdem directione pergentia vel ſunt convexa vel concavo-convexa inter ſe, & ex eâdem vel diverſâ aſymptoti parte in infinitum progrediuntur hyperbolica crura. E caſus quarti parte ſecundâ conſtat ſpecies hujus æquationis haud plures eſſe quam 4×19 ſine nodo vel cuſpide ad curvam vel interſectione duarum curvarum;

rum; haud plures eſſe, quam 4×8 nodum vel cuſpidem ad hanc vel illam curvam habentes, haud plures quam $4 \times 2 \times 3$ duos nodos vel nodum & cuſpidem ad idem punctum curvæ, vel 4×3 duos nodòs ad diverſas curvas poſitos, vel 36 nodum & cuſpidem vel duas cuſpides habentes, vel 4×2 tres nodos ad idem punctum curvæ vel duos nodos & cuſpidem, vel 4×2 nodum vel cuſpidem ad unam curvam & nodum & cuſpidem vel duos nodos ad alteram, vel plures quam 4×28, quarum duæ curvæ ſeſe interſecant; vel 4×7 in quibus curvæ bis ſeſe interſecant; vel 4 in quibus curvæ ter ſeſe interſecant ergo totus numerus ſpecierum hujus caſus haud major eſt quam 492.

CAS. 9.

1. Sit æquatio $x \times \overline{x+a} \times y^2 = -Ax^4 + Bx^3 + Cx^2 + Dx + E$, & deſignat hyperbolas duas parallelas aſymptotos & diametrum habentes.

Earum ſpecies ſunt, vel duæ puræ inſcriptæ hyperbolæ inter aſymptotos jacentes, vel hæ duæ hyperbolæ cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto, vel ovali nodum vel cuſpidem habente, vel duobus punctis, vel ex unâ aſymptotum parte ovali vel puncto & ovali vel puncto ex alterâ aſymptotum parte; vel duæ conchoidales hyperbolæ extra aſymptotos, vel hæ duæ hyperbolæ cum ovali vel puncto inter hyperbolas, necne, vel nodo vel cuſpide; vel una conchoidalis inter aſymptotos, altera vero extra eas, vel hæ duæ hyperbolæ cum ovali vel puncto inter aſymptotos vel ex hâc vel illâ aſymptotum parte, vel cum nodo vel cuſpide ad hanc vel illam hyperbolam; vel duæ conchoidales hyperbolæ inter aſymptotos, vel hæ cum ovali vel puncto inter vel extra eas, vel nodo vel cuſpide, vel hæ duæ hyperbolæ ſeſe interſecant figurâ cruciformi, vel ſeſe interſecant & habent ovalem vel punctum, vel bis ſeſe interſecant, ergo numerus ſpecierum eſt 40.

2. Sit æquatio

$$x \times \overline{x+a}\, y^2 + \overline{a+bx+cx^2+dx^3} \times y = Ax^4 + Bx^3 + Cx^2 + Dx + E,$$

& radices æquationis $y^2 + dy = A$ impoſſibiles, & deſignat curvas duas parallelas aſymptotos habentes.

Earum ſpecies ſunt, vel duæ anguineæ hyperbolæ quarum duæ ſunt ſpecies; vel hæ hyperbolæ cum ovali vel puncto vel nodo vel cuſpide inter vel extra aſymptotos; vel cruciformes fiant anguineæ, vel habent etiam ovalem vel punctum vel nodum vel cuſpidem, vel bis cruciformiter ſeſe interſecant; vel duæ ſunt ambigenæ hyperbolæ, vel hæ hyperbolæ cum ovali vel puncto vel nodo vel cuſpide vel duabus ovalibus, vel ovali & puncto, vel ovali nodum vel cuſpidem habente, vel duobus punctis, vel nodo vel cuſpide ad curvam cum ovali vel puncto; vel ad curvam duobus nodis vel nodo & cuſpide; vel curva habet ovalem vel punctum, vel nodum, vel cuſpidem ex unâ aſymptotum parte, & ovalem, vel punctum, vel nodum vel cuſpidem, ex alterâ aſymptotum parte; vel una eſt hyperbola inſcripta & altera circumſcripta, vel hæ cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto, vel ovali nodum vel cuſpidem habente, vel duobus punctis, vel cum nodo ad curvam etiamque cum ovali vel puncto vel nodo vel cuſpide ex hâc vel illâ aſymptotum parte; & ſic de cuſpide ad curvam; vel eſt una curva inſcripta & altera ambigena; vel hæ hyperbolæ habent etiam ovalem vel punctum ex hâc vel illâ aſymptotum parte vel nodum vel cuſpidem, vel duas ovales, vel ovalem & punctum, vel duo puncta ex hâc vel illâ ex eâdem vel diverſâ aſymptotum parte, vel ovalem nodum vel cuſpidem habentem ex hâc vel illâ aſymptotum parte, vel ambigena curva habet nodum vel cuſpidem & ovalem vel punctum ex eâdem vel diverſâ aſymptotum parte, vel habet duos nodos vel nodum & cuſpidem; vel eſt conchoidalis hyperbola, quæ ponitur vel inter vel extra aſymptotos & anguinea hyperbola, vel hæ habent inter eas ovalem vel punctum vel ad anguineam vel conchoidalem nodum vel cuſpidem, vel extra aſymptotos ovalem vel punctum ex hâc vel illâ aſymptotum parte, vel ad anguineam vel conchoidalem nodum vel cuſpidem, vel curvæ fiant cruciformes, vel ſint cruciformes & habeant extra eas ovalem vel punctum ex hâc vel illâ aſymptotum parte, vel ad anguineam nodum vel cuſpidem; vel bis fiant cruciformes; his denique ſpeciebus adjiciendæ ſunt tot ſpecies, quot habet prior æquatio, & erit numerus ſpecierum $40+147=187$.

3. Sit

3. Sit æquatio $x^2y^2+\overline{a+bx}\times y=Ax+B$; & deſignat hyperbolas duas aſymptotos habentes; curvarum ſpecies ſunt vel, duæ hyperbolæ, una inſcripta & altera circumſcripta circa eundem angulum, vel hæ duæ hyperbolæ cum ovali vel puncto vel nodo vel cuſpide, vel duæ convergentes hyperbolæ; vel hæ duæ hyperbolæ ſeſe interſecant figurâ cruciformi.

Si terminus (Ax) deeſt, tum quatuor hyperbolæ ſunt circa duas aſymptotos deſcriptæ.

Sit æquatio $x^2y^2=Ax+B$; & curva habet diametrum, quarum ſpecierum numerus eſt 9.

4. Sit æquatio

$$xy^3+\overline{a+bx+c}\,x^2\times y^2+\overline{e+fx+gx^2+px^3}\times y=Ax^2+Bx+C$$

in quâ radices æquationis $y^2+cy+b=0$ ſunt impoſſibiles, & deſignat duas hyperbolas.

Hæ autem hyperbolæ ſunt vel una inſcripta, ambigena vel circumſcripta, & altera etiam vel inſcripta ambigena vel circumſcripta.

Et ſpecierum numerus haud major eſſe poteſt, quam ut conſtat e parte ſecundâ caſus 4[ti] 9×23 ſine nodo, cuſpide vel ad curvam cruciformi, non plures quam $15\times\overline{9+8}$ unum nodum vel cuſpidem ſolummodo habentes, non plures quam 15×3 duos nodos vel nodum & cuſpidem ad idem vel diverſum curvæ punctum, vel duas cuſpides; vel 15 tres cuſpides, vel nodum & duas cuſpides, quarum una vel ponitur ad idem vel diverſum punctum curvæ; vel duos nodos ad idem vel diverſum curvæ punctum & cuſpidem ad idem vel diverſum punctum; non plures quam 9×3 nodum vel cuſpidem ad utraſque curvas habentes, non plures quam 15×3 nodum ad unam & cuſpidem ad alteram curvam habentes, vel 15 nodum & cuſpidem ad unam & cuſpidem ad alteram, non plures ſunt quam 270, curvæ, quarum crura ſemel ſolummodo ſeſe ſecant cruciformiter; non plures quam 87, in quibus curvæ bis ſecant ſeſe cruciformiter, non plures quam 9 in quibus ter ſeſe ſecant cruciformiter, ergo totus numerus ſpecierum hujus caſus haud major eſt quam $1254+9+187=1450$.

CAS.

CAS. 10.

Sit $x\times\overline{x+a}\times y^2+\overline{ex+f}\times y=Ax+B$, & hyperbolas deſignat tres aſymptotos habentes, quarum duæ ſunt inter ſe parallelæ.

Earum ſpecies ſunt vel inſcripta, ambigena & anguinea hyperbola, cujus duæ ſunt ſpecies, vel ambigena & anguinea ſeſe interſecant, vel una inſcripta cum duabus ambigenis; vel hæ cum ovali vel puncto vel nodo vel cuſpide; vel una hyperbola ſerpens circa ejus aſymptoton in infinitum cum duabus ambigenis, vel ſerpens & ambigena ſeſe interſecant, vel duæ inſcriptæ hyperbolæ, quarum una ponitur inter duas parallelas aſymptotos, & tertia circumſcripta; vel hæ habent ovalem vel punctum vel ad circumſcriptum nodum vel cuſpidem.

Si terminus (fy) deſit, tum vel datur hyperbola in infinitum circa ejus aſymptoton ſerpens, & inſcripta & ambigena, quarum duæ ſunt ſpecies, vel ambigena & ſerpens hyperbola ſeſe interſecant, vel ſint duæ inſcriptæ hyperbolæ cum anguineâ, vel inſcripta ambigena & conchoidalis hyperbola inter vel extra aſymptotos, ſi ponitur conchoidalis inter aſymptotos, ambigena & conchoidalis ſeſe interſecare poſſunt; vel ſint duæ inſcriptæ & ambigena hyperbola, vel hæ cum ovali vel puncto.

Sit $x\times\overline{x+a}\times y^2+\overline{ex+f}\times y=B$; & vel duæ ſunt inſcriptæ hyperbolæ & duæ ambigenæ, quarum duæ ſunt ſpecies, vel duæ ambigenæ ſeſe interſecant, vel tres inſcriptæ & una ambigena hyperbola, & idem erit numerus ſpecierum, ſi utræque radices æquationis $y^2+ey=B$ ſunt vel affirmativæ vel negativæ.

Sit $x\times\overline{x+a}\times y^2=Ax+B$, & curva habet diametrum, ſpecies earum ſunt, vel duæ inſcriptæ hyperbolæ cum conchoidali vel inter vel extra aſymptotos, vel una hyperbola ſerpens circa ejus aſymptoton in infinitum cum duabus inter parallelas aſymptotos inſcriptis; vel quatuor inſcriptæ hyperbolæ, ergo numerus ſpecierum eſt 38.

Sit æquatio $x^2y^2+\overline{x^3+bx^2+cx+d}\times y=Ax^2+Bx+C$, & denotat hyperbolas tres aſymptotos habentes.

Earum ſpecies ſunt vel una hyperbola inſcripta, ambigena, vel circumſcripta; & hyperbola, quæ ei opponitur, vel inſcripta, ambigena, vel circumſcripta; & tertia convergens hyperbola; vel hæ cum figurâ cruciformi, ex unâ, alterâ, vel utrâque parte vel etiam ovali, vel

vel puncto vel nodo vel cuſpide, &c. ſi terminus (dy) deſit, tum transformatur convergens hyperbola vel in duas conchoidales, circa eandem aſymptoton hyperbolas, vel in duas, quarum reſpectiva crura eodem aſymptoti latere & eadem directione in infinitum ſerpunt, harum ſpecierum numerus eſt 216.

Si nullæ ſint oppoſitæ hyperbolæ, tum hyperbolæ vel deſcribuntur ad angulos vel latera aſymptotum, & ſpecies ſunt vel una inſcripta vel ambigena, altera ambigena vel circumſcripta & tertia inſcripta ambigena vel circumſcripta hyperbola, vel hæ cum ovali vel puncto; vel nodo vel cuſpide ad hanc vel illam curvam; vel cum duabus ovalibus, vel ovali & puncto vel nodo vel cuſpide, vel duobus punctis, vel nodo vel cuſpide ad curvam cum ovali vel puncto; vel duobus nodis vel nodo & cuſpide ad eandem vel diverſam curvam, vel duabus cuſpidibus; vel ſemel bis vel ter cruciformes fiant eædem hyperbolæ, vel ſit etiam conchoidalis hyperbola, & curvæ habent ovalem vel punctum vel nodum vel cuſpidem, vel unam vel duas interſectiones, vel quatuor ſunt hyperbolæ, ergo totus numerus ſpecierum hujus caſus eſt 1263.

CAS. II.

Sit æquatio $xy^2 + \overline{ax^3 + bx^2 + cx + d} \times y = Ax^2 + Bx + C$, & deſignat curvas duas hyperbolas & parabolam conici generis habentes.

Earum ſpecies ſunt vel duæ curvæ partim parabolicæ & partim hyperbolicæ cum anguinea vel conchoidali hyperbola, vel hæ curvæ cruciformiter ſeſe interſecant ex una vel utrâque parte; vel habeant ovalem vel punctum vel ad unam e curvis nodum vel cuſpidem, vel duplicem interſectionem, vel duplicem interſectionem ad unam curvam & interſectionem ad alteram; quarum ſpecierum numerus eſt 134; vel duæ curvæ partim parabolicæ & partim hyperbolicæ cum ambigenâ vel inſcriptâ vel circumſcriptâ hyperbolâ, vel hæ ſeſe interſecant, vel habent ovalem vel punctum vel duas ovales, vel ovalem & punctum vel ovalem nodatam vel cuſpidatam vel duo puncta, vel nodum vel cuſpidem ad unam e curvis, vel nodum vel cuſpidem cum ovali vel puncto, vel duos nodos vel nodum & cuſpidem ad eandem vel diverſas curvas, vel duas cuſpides, vel unam vel duas vel tres interſectiones, quarum ſpecierum numerus eſt 290.

2. Sit

2. Sit æquatio

$$xy^3+\overline{2ax^2+bx+c}\times y^2+\overline{a^2x^3+dx^2+ex+f}\times y=Ax^2+Bx+C$$

& denotare poteſt curvas duas hyperbolas & parabolam conici generis habentes.

Earum ſpecies ſunt vel parabola & duæ hyperbolæ, vel duæ curvæ partim parabolicæ & partim hyperbolicæ cum inſcriptâ vel ambigena vel circumſcriptâ hyperbolâ; numerus autem harum curvarum ſpecierum ſine nodo, cuſpide vel cruciformi ad curvam non major erit quam 19×33; numerus autem cum nodo vel cuſpide, nodis vel cuſpidibus ſine cruciformi non major erit, quam 120×33, cum vero cruciformi non major erit quam 195×33, ergo totus numerus curvarum hujus caſus non major erit quam 8342.

CAS. 12.

1. Sit æquatio

$$xy^3+\overline{a+bx+cx^2+ex^2}\times y^2+\overline{d+ex+fx^3+gx^3}\times y=Ax^2+Bx+C$$

& deſignare poteſt curvas quatuor aſymptotos habentes.

Hyperbolæ vel jacent ad duo latera trapezii ab aſymptotis, comprehenſi & duos ejus angulos, vel ad tria latera & unum angulum.

Et poſſunt eſſe quatuor hyperbolæ vel inſcriptæ, ambigenæ vel circumſcriptæ, quarum ſpecierum numerus eſt 2×49=98.

Numerus curvarum ſine nodo, cuſpide vel cruciformi non major eſt quam 19×98=1862.

Numerus curvarum cum nodo vel cuſpide vel duobus nodis, vel nodo & cuſpide, vel duabus cuſpidibus, vel tribus nodis, vel duobus nodis & cuſpide, vel nodo & duabus cuſpidibus, vel tribus cuſpidibus ſine cruciformi, haud major erit, quam 98×212=20776.

Numerus curvarum cum cruciformi vel duabus vel tribus cruciformibus & nodo vel nodis, cuſpide vel cuſpidibus haud major eſt, quam 98×376=36848.

Vel una e quatuor hyperbolis continere poteſt in ſe alteram, vel tres ſint hyperbolæ & anguinea vel conchoidalis; & numerus harum curvarum ſine nodo vel cuſpide, &c. non major eſt, quam 864.

Numerus curvarum cum nodo vel cuſpide, &c. ſine cruciformi non major erit, quam 1728.

Numerus curvarum ſeſe interſecantium cruciformiter non major erit, quam 864.

2. Si

2. Si curvæ habent generale centrum, i. e. quatuor aſymptoti ſeſe interſecant in eodem puncto; numerus curvarum non major eſt quam 2500.

3. Sit æquatio $y^4+\overline{a+bx+cx^2}\times y^2=Ax^4+Bx^3+Cx^2+Dx+E$, & deſignat curvas diametrum habentes.

Curvæ collocantur ad angulos & ad latera figuræ ab interſectione aſymptotum formatæ ut in caſu priori, & ſunt vel duæ inſcriptæ, ambigenæ, vel circumſcriptæ hyperbolæ & duæ aliæ etiam vel inſcriptæ, ambigenæ, vel circumſcriptæ hyperbolæ, harum ſpecierum numerus non major eſt quam 2392. Si deſint termini bxy^2 & Bx^3 curvæ habent generale centrum, & earum numerus ut conſtat e præcedenti non major erit quam 132.

Si $y^4\overline{+a+bx^2}\times y^2=Ax^4+Cx^2+E$; & curvæ habent duas vel quatuor diametros, & numerus earum ſpecierum non major erit quam 42.

4. Sit æquatio $x\times\overline{x+a}\times y^2+\overline{ax^3+bx^2+cx+d}\times y=Ax^2+Bx+C$, & denotat hyperbolas quatuor aſymptotos habentes, quarum duæ ſunt inter ſe parallelæ, quarum ſpecies ſunt vel una inſcripta, ambigena, vel circumſcripta; & altera etiam vel inſcripta ambigena vel circumſcripta cum duabus ambigenis vel anguineis vel unâ inſcriptâ & alterâ ambigenâ, vel unâ conchoidali & alterâ anguinea, &c. quarum ſpecierum numerus erit 4097.

Sit æquatio $x\times\overline{x+a}\times y^2=+Ax^4+Bx^3+Cx^2+Dx+E$, & denotat hyperbolas diametrum & quatuor aſymptotos habentes, quarum duæ ſunt inter ſe parallelæ, & numerus harum curvarum ut facile colligi poteſt, erit 163. & numerus curvarum duas aſymptotos inter ſe parallelas & duas alias inter ſe parallelas habentium ſine diametro non major erit quam 193, cum vero unâ diametro eſt 6, & duabus diametris 3.

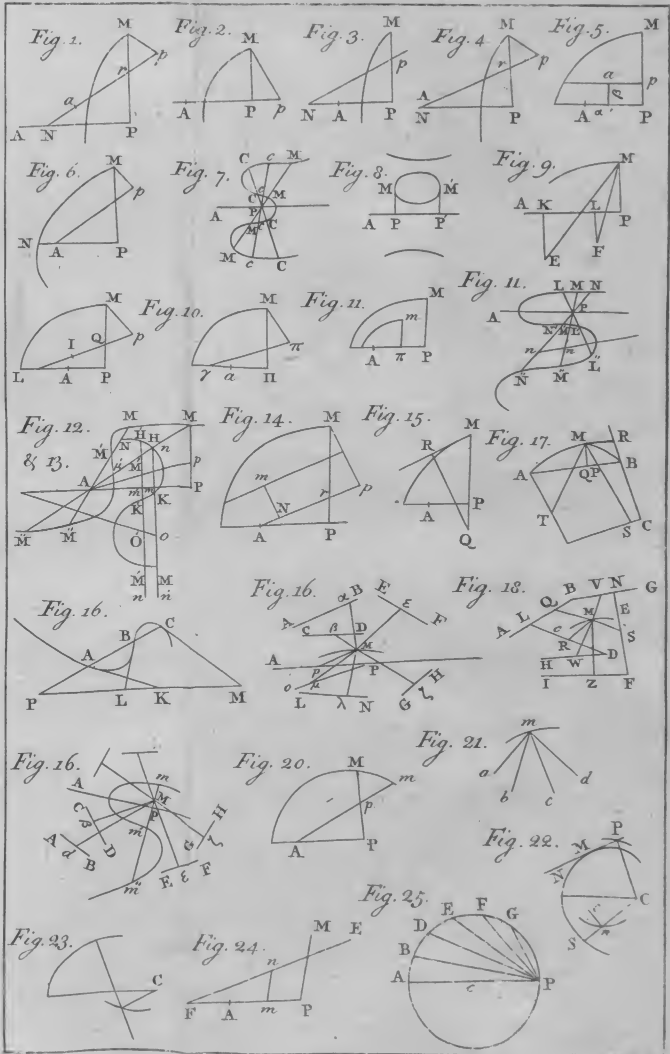
5. Sit æquatio $\overline{x^3+ax^2+bx+c}\times y=Ax^2+Bx+C$, & deſignare poteſt decem curvas quatuor aſymptotos habentes, quarum tres ſunt inter ſe parallelæ; ergo totus numerus curvarum hujus caſus non erit major quam 72480. & non plures ſunt quam 84551. curvæ tertii ordinis.

Si abſciſſa incipit a puncto duplici, tum deſunt termini $\alpha x+\beta y+K$; ſi incipit a puncto triplici, tum deſunt termini $\gamma x^2+\delta xy+\varepsilon y^2+\alpha x+\beta y+K$; ſi incipit a puncto quadruplici, tum deſunt termini $Zx^3+nx^2y+\Theta xy^3+\kappa y^3+\gamma x^2+\delta xy+\varepsilon y^2+\alpha x+\beta y+K$; & ſic deinceps.

FINIS.

CORRIGENDA.

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Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.
Fig. 11.
Fig. 12. & 13.
Fig. 14.
Fig. 15.
Fig. 17.
Fig. 16.
Fig. 16.
Fig. 18.
Fig. 16.
Fig. 20.
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Fig. 24.

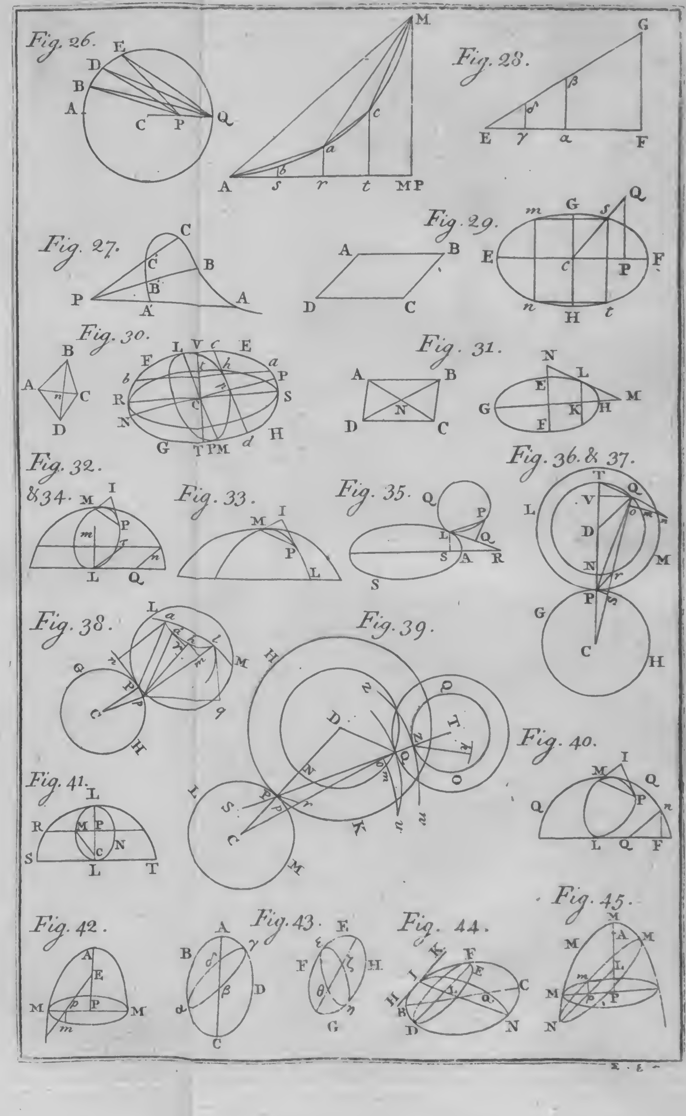


Fig. 26.
Fig. 28.
Fig. 27.
Fig. 29.
Fig. 30.
Fig. 31.
Fig. 32. &34.
Fig. 33.
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Fig. 36. & 37.
Fig. 38.
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Fig. 40.
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Fig. 43.
Fig. 44.
Fig. 45.

ACCADE[illegible]
R. DELLE S[illegible]
DI TORINO

Fig. 46.

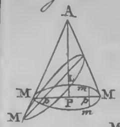


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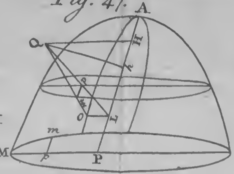


Fig. 48.

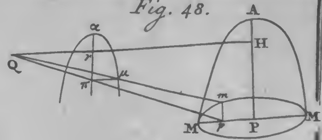


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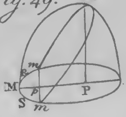


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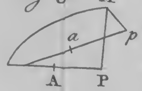


Fig. 51.

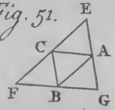


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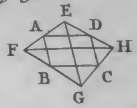


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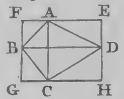


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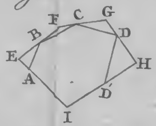


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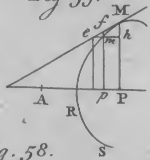


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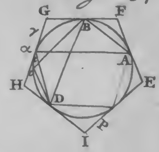


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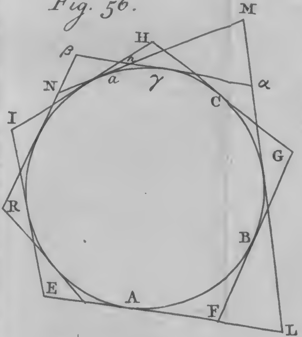


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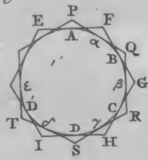


Fig. 57.

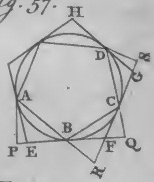


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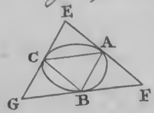


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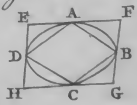


Fig. 61.

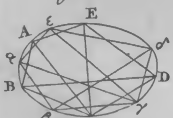


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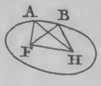


Fig. 63.

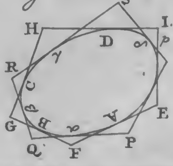
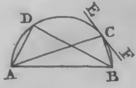


Fig. 64.



ACCADEMIA
DELLE SCIENZE
TORINO